

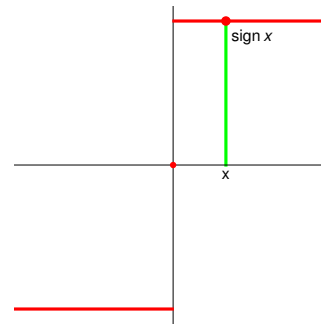
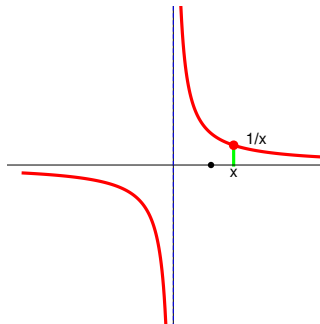
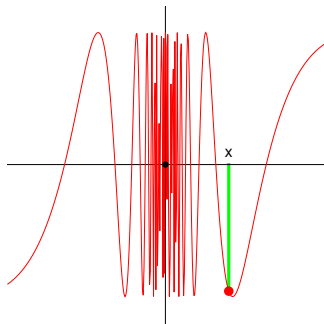
# INTRODUCTION TO CALCULUS

MATH 1A

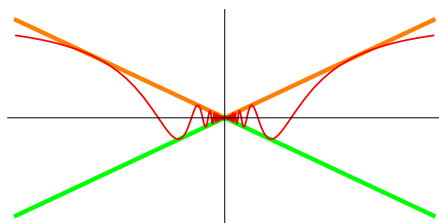
## UNIT 5: WORKSHEET

There are a few mechanisms for discontinuity . A function can **jump for good**, **badly rush to infinity** or have an **ugly oscillation**. All three cases come from a division by zero somewhere.

| Nice Guys                            | Good,Bad,Ugly Guys             |
|--------------------------------------|--------------------------------|
| $x^2 + 4x + 6$                       | $1/x$ at 0                     |
| $\sin(x), \cos(x)$                   | $\tan(x)$ at $\pi/2$           |
| $\exp(x)$                            | $\log x $ at 0                 |
| $\text{sinc}(x) = \frac{\sin(x)}{x}$ | $\frac{1}{\cos(x)}$ at $\pi/2$ |



## Squeeze theorem



Lets look at the **tamed devil**  $f(x) = x \sin(\frac{1}{x})$  . A bad oscillation is tamed by the **squeeze theorem**: We have  $-|x| \leq x \sin(\frac{1}{x}) \leq |x|$  and both  $g(x) = |x|$  and  $h(x) = -|x|$  are continuous at 0 and have the same value at 0. Because  $f$  is sandwiched between two continuous functions which come together at the point, the function must be continuous and give that common value.

## Which functions are continuous?

If you can assign a value at a point where it is not defined so that it becomes overall continuous, we just consider this continuous. We consider  $(x^4 - 1)/(x - 1)$  to be continuous for example because it is equivalent for  $x \neq 1$  to  $x^3 + x^2 + x + 1$  which is continuous. We consider  $\sin(x)/x$  to be continuous because we can fill in a value 1 at  $x = 0$  to make it continuous overall. We saw that  $x \sin(x)$  is continuous everywhere because of the squeeze theorem. Which of the following functions are continuous?

**Problem 1:**  $f(x) = x^2 + x^2 \sin(1/x^2)$

### Solution:

Yes, the function is continuous with the understanding  $f(0) = 0$ . The only point to fix is  $x = 0$ . By the squeeze theorem, we can assign a value 0 at 0.

**Problem 2:**  $f(x) = \sqrt{|x|}$

### Solution:

This function is  $\sqrt{x}$  for  $x \geq 0$  which is continuous. The absolute value fixes it also for  $x < 0$ . The function is continuous overall.

**Problem 3:**  $f(x) = |x^3|/x^3$

### Solution:

This solution is equal to 1 for  $x > 0$  and  $-1$  for  $x < 0$ . It can not be fixed at  $x = 0$ . There is a jump discontinuity there. The function is not continuous.

**Problem 4:**  $f(x) = |x|^2/x$

**Solution:**

Since  $|x|^2 = x^2$ , we can divide this by  $x$  and get  $x$ . This function is continuous. While the original expression  $|x|^2/x$  was not defined at  $x = 0$ , we can find a value  $f(0) = 0$  to render the function continuous.

**Problem 5:**  $f(x) = \frac{1}{\sqrt{|x|}}$

**Solution:**

This function blows up at  $x = 0$ . There is no way we can assign a finite value at  $x = 0$ . The function is not continuous.

**Problem 6:**  $\frac{1}{\log|1/x|}$

**Solution:**

We see that at  $x = 1$  we have  $\log|1/x| = 0$  so that we divide by 0 there. The function is not continuous.

**Problem 7:**  $\log(\log|x|)$

**Solution:**

This function goes to  $\infty$  both at  $x = 0$  and  $x = 1$ . The function is actually not defined even for  $x \in (-1, 1)$ . It is not continuous on  $[1, \infty)$  nor on  $(-\infty, 1]$ .

**Problem 8:**  $1/(1 + |x|)$

**Solution:**

This function is continuous everywhere because  $1 + |x|$  is continuous and positive everywhere.

**Problem 9:**  $1/(1 - |x|)$

**Solution:**

This function has a pole at  $x = 1$  and  $x = -1$ . It is not continuous.

**Problem 10:**  $x^2/\sin(x)$

**Solution:**

This function can be written as  $x(x/\sin(x))$ . Both functions  $f(x) = x$  and  $g(x) = x/\sin(x)$  are continuous everywhere. So, also the product is at 0. However, there is a catch (which is a bit sneaky). The function is not continuous because at  $x = \pi$  we have nothing to save us. The function is not continuous.