INTRODUCTION TO CALCULUS

MATH 1A

Unit 3: Worksheet

Problem 1): **Diana Nyad** in 2013 became the first person to swim the 110-miles from Cuba to Florida without a shark cage. We just watched clip from the movie that had been released in 2023 and was directed by Jimmy Chin. There is an interesting data riddle involved in the case of Nyad's swim. Nyad released GPS data about her 53-hour swim. Here's a very small sample of the data:

Time (hours)	0	5	7	15	31	32	40
Distance (miles)	0	6.67	8.90	18.72	54.85	58.72	82.88

Find Nyad's average speed between 5 and 7 hours into her swim. Also find her average speed between 31 and 32 hours into her swim. Why might these numbers have raised suspicion?

Solution:
The average speed over an interval is equal to $\frac{\text{distance}}{\text{time}}$. Between 5 and 7 hours, this is $\frac{8.90 \text{ mi.} - 6.67 \text{ mi.}}{7 \text{ h} - 5 \text{ h}} = \boxed{1.115 \text{ mph}}$. Between 31 and 32 hours, the average speed is $\frac{58.72 \text{ mi.} - 54.85 \text{ mi.}}{32 \text{ h} - 31 \text{ h}} = \boxed{3.87 \text{ mph}}$. Nyad more than tripled her speed after she had been swimming for 31 hours. There were however strong currents in her favor. The movie illustrates this.

Problem 2): What can you say about tangent lines in the following three cases.

- When a curve is concave up.
- When a curve is concave down.
- When a curve is a straight line.

Solution:
If the curve is concave up the tangent lines are below the curve. If the curve is concave down, the tangent lines are above the curve. If the curve is a straight line, the tangent line agrees with the curve.

Problem 3): Lets look at the Fibonacci numbers (we had no time for them in the first lecture). The function values are given on the integers $f(0) = 0, f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 3, f(5) = 5, f(6) = 8, f(7) = 13, f(8) = 21$.
First find a formula for the rate of change $f(x+1) - f(x)$.

P.S. You might be surprised to learn that there is an explicit function $f(x)$ which gives us these Fibonacci data. It is

$$f(x) = \frac{\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^x}{\sqrt{5}} - \frac{\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^x}{\sqrt{5}} .$$

Solution:
If you compute $f(x+1) - f(x)$ you get $f(x-1)$.

Problem 4): Draw the graph of $f(x) = 9 - 4(x-2)^2$, and put the following quantities in ascending order (from smallest to largest), with "<" or "=" signs between them.

(1) The slope of the secant line between $x = 1$ and $x = 2$. (This is the average rate of change on $[1, 2]$).

(2) The slope of the tangent line at $x = 0$.
(3) The slope of the tangent line at $x = 1$.
(4) $\dfrac{f(2.5) - f(1)}{1.5}$

Solution:
To get the graph $y = 9 - 4(x-2)^2$, we can start with $y = x^2$, shift it right by 2 to get $y = (x-2)^2$, vertically flip it and stretch it by a factor of 4 to get $y = -4(x-2)^2$, and move it up by 9 to get $y = 9 - 4(x-2)^2$: Then, we can picture the given quantities as slopes on the graph of f. In particular, $\dfrac{f(2.5) - f(1)}{1.5}$ can be thought of as the slope of the secant line between $(1, f(1))$ and $(2.5, f(2.5))$. So, here are all of the slopes on the graph: From this, we see that $\boxed{\text{D} < \text{A} < \text{C} < \text{B}}$.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024