INTRODUCTION TO CALCULUS

MATH 1A

Unit 2: Worksheet

Problem 1): The cosine and sine functions can be defined geometrically by the coordinates $(\cos(x), \sin(x))$ of a point on the unit circle. The tangent function is defined as $\tan(x) = \sin(x)/\cos(x)$. Define also $\cot(x) = 1/\tan(x)$. [1]

$\sin(x) =$ opposite/hypothenuse (SOH)
$\cos(x) =$ adjacent/hypothenuse (CAH)
$\tan(x) =$ opposite/adjadcent (TOA)

Problem a): Find $\cos(\pi/3)$, $\sin(\pi/3)$.

Problem b): Find the roots of $f(x) = \cos(x)$ and $f(x) = \sin(x)$.

Problem c): Find the roots of $f(x) = \cos(x-1)$ and $f(x) = \sin(x+1)$.

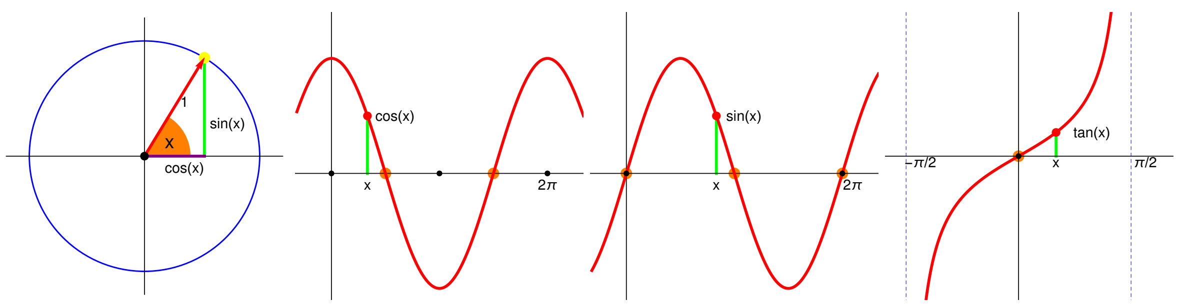
Problem d): Find $\tan(3\pi/2)$ and $\cot(3\pi/2)$.

Problem e): Find $\cos(3\pi/2)$ and $\sin(3\pi/2)$.

Problem f): Find $\tan(\pi/4)$ and $\cot(\pi/4)$.

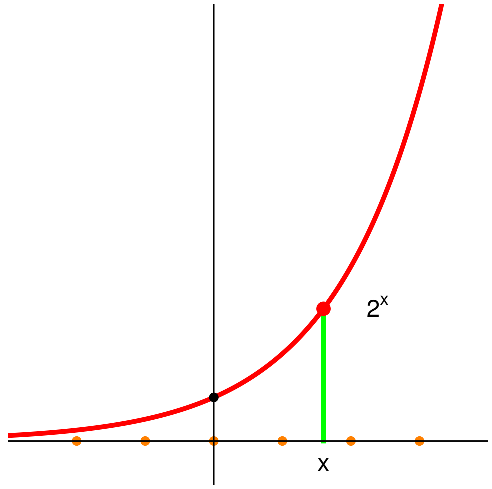
We have $\cos(\pi/2 - x) = \sin(x)$ and $\sin(\pi/2 - x) = \cos(x)$. **Pythagoras** gives:

$$\boxed{\cos^2(x) + \sin^2(x) = 1}$$



Problem 2): The function $f(x) = 2^x$ is first defined for positive integers like $2^{10} = 1024$, then for all integers with $f(0) = 1, f(-n) = 1/f(n)$. By taking roots, one can define $2^{3/2} = 8^{1/2} = \sqrt{8} = 2.828...$. Since the function 2^x is monotonone on the set of rationals, we can fill the gaps and define $f(x)$ for any real x. By taking square roots again and again for example, we see $2^{1/2}, 2^{1/4}, 2^{1/8}, ...$ we approach $2^0 = 1$.

[1] One uses abbreviations $\sec(x) = 1/\cos(x), \csc(x) = 1/\sin(x)$.



Problem a):	What is 2^{-5}?
Problem b):	Find $2^{1/2}$.
Problem c):	Find $27^{1/3}$.
Problem d):	Find $64^{-1/4}$.
Problem e):	What is larger: $A = 2^{3/4}$ or $B = 2^{4/5}$?
Problem f):	What is larger: $A = 2^{-1/2}$ or $B = 2^{-1/3}$?

There is nothing special about 2 and we can take any positive base a and define the exponential a^x. It satisfies $a^0 = 1$ and the remarkable rule:

$$a^{x+y} = a^x \cdot a^y$$

The exponential function is spectacular because it gives a link between addition and multiplication. Lets look next at the definition of the exponential function.

First define

$$\exp_h(x) = (1+h)^{x/h}$$

where h is a positive parameter. It satisfies

$$\exp_h(x+h) = (1+h)\exp_h(x)$$

so that

$$[\exp_h(x+h) - \exp_h(x)]/h = \exp_h(x) \; .$$

For $h = 1$, we have the function 2^x. It satisfies $2^{x+1} - 2^x = 2^x$. In the limit $h \to 0$, we get the **exponential function** $\exp(x)$ which we call e^x. Evaluated at $x = 1$ gives the **Euler number** $e = e^1 = 2.71828\ldots$.

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