INTRODUCTION TO CALCULUS

MATH 1A

Unit 2: Worksheet

Problem 1): The cosine and sine functions can be defined geometrically by the coordinates $(\cos(x), \sin(x))$ of a point on the unit circle. The tangent function is defined as $\tan(x) = \sin(x)/\cos(x)$. Define also $\cot(x) = 1/\tan(x)$. [1]

$\sin(x) =$ opposite/hypothenuse (SOH)
$\cos(x) =$ adjacent/hypothenuse (CAH)
$\tan(x) =$ opposite/adjadcent (TOA)

Problem a): Find $\cos(\pi/3)$, $\sin(\pi/3)$.

Problem b): Find the roots of $f(x) = \cos(x)$ and $f(x) = \sin(x)$.

Problem c): Find the roots of $f(x) = \cos(x - 1)$ and $f(x) = \sin(x + 1)$.

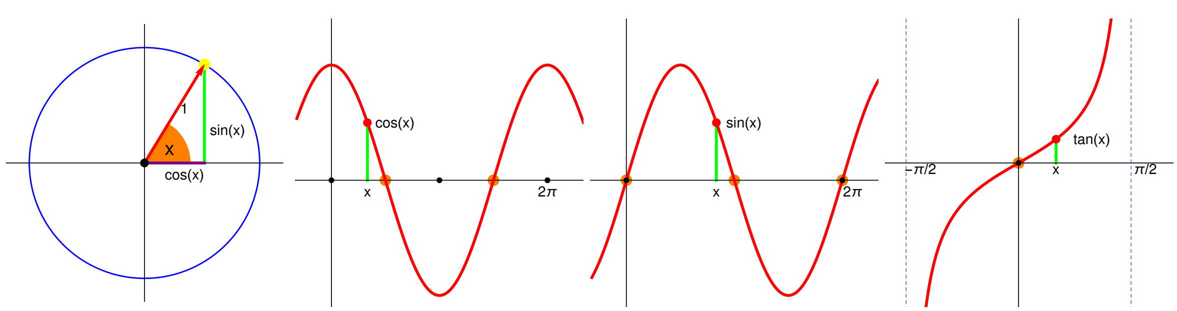
Problem d): Find $\tan(3\pi/2)$ and $\cot(3\pi/2)$.

Problem e): Find $\cos(3\pi/2)$ and $\sin(3\pi/2)$.

Problem f): Find $\tan(\pi/4)$ and $\cot(\pi/4)$.

We have $\cos(\pi/2 - x) = \sin(x)$ and $\sin(\pi/2 - x) = \cos(x)$. **Pythagoras** gives:

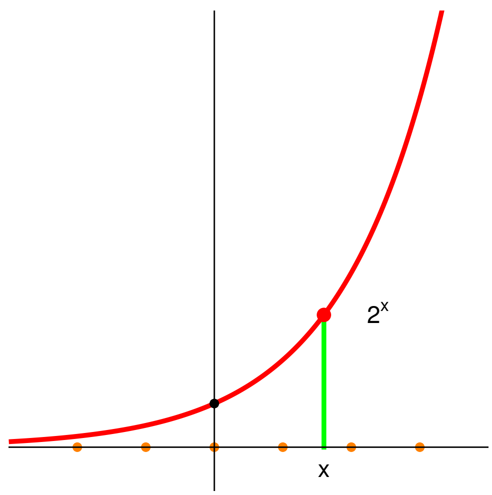
$$\boxed{\cos^2(x) + \sin^2(x) = 1}$$



[1] One uses abbreviations $\sec(x) = 1/\cos(x), \csc(x) = 1/\sin(x)$.

Solution:
a) $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$.
b) $\cos(\pi/2 + k\pi) = 0$, $\sin(k\pi) = 0$, where k is an integer.
c) Just shifted $1 + \pi/2 + k\pi$ and $-1 + k\pi$ are the roots, where k is an integer.
d) $\tan(3\pi/2)$ is the same than $\tan(\pi/2$ and is undefined. $\cot(3\pi/2)$ is the same than $\cot(\pi/2)$ which is zero.
e) Both $\tan(\pi/4)$ and $\cot(\pi/4)$ are 1.

Problem 2): The function $f(x) = 2^x$ is first defined for positive integers like $2^{10} = 1024$, then for all integers with $f(0) = 1, f(-n) = 1/f(n)$. By taking roots, one can define $2^{3/2} = 8^{1/2} = \sqrt{8} = 2.828....$ Since the function 2^x is monotonone on the set of rationals, we can fill the gaps and define $f(x)$ for any real x. By taking square roots again and again for example, we see $2^{1/2}, 2^{1/4}, 2^{1/8}, ...$ we approach $2^0 = 1$.



Problem a): What is 2^{-5}?

Problem b): Find $2^{1/2}$.

Problem c): Find $27^{1/3}$.

Problem d): Find $64^{-1/4}$.

Problem e): What is larger: $A = 2^{3/4}$ or $B = 2^{4/5}$?

Problem f): What is larger: $A = 2^{-1/2}$ or $B = 2^{-1/3}$?

Solution:
a) 1/32.
b) $\sqrt{2}$
c) 3
d) 1/4
e) $2^{0.8}$ is larger than $2^{0.75}$.
f) Since 1/2 is larger than 1/3 so that $2^{1/2}$ is larger than $2^{1/3}$. Therefore $2^{-1/2} = 1/2^{1/2}$ is smaller than $2^{-1/3} = 1/2^{1/3}$.

There is nothing special about 2 and we can take any positive base a and define the exponential a^x. It satisfies $a^0 = 1$ and the remarkable rule:

$$a^{x+y} = a^x \cdot a^y$$

The exponential function is spectacular because it gives a link between addition and multiplication. Lets look next at the definition of the exponential function.

First define

$$\exp_h(x) = (1+h)^{x/h}$$

where h is a positive parameter. It satisfies

$$\exp_h(x+h) = (1+h)\exp_h(x)$$

so that

$$[\exp_h(x+h) - \exp_h(x)]/h = \exp_h(x)\ .$$

For $h = 1$, we have the function 2^x. It satisfies $2^{x+1} - 2^x = 2^x$. In the limit $h \to 0$, we get the **exponential function** $\exp(x)$ which we call e^x. Evaluated at $x = 1$ gives the **Euler number** $e = e^1 = 2.71828\ldots$.

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