INTRODUCTION TO CALCULUS

MATH 1A

Unit 34: Music

Music is a function

34.1. A piece of music is a pair of **functions** $f(t)$ and $g(t)$. They tell the displacement of the membrane of the left and right loudspeakers. This motion produces sound waves that reach your ear. In Mathematica, we can play a function by replacing "Plot" with "Play". For example:

```
Play[ Sin[2Pi 1000 x^2],{x,0,10}]
```

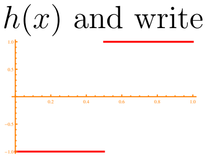
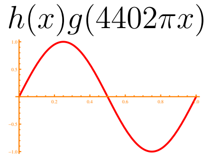
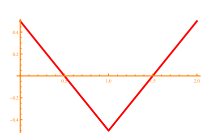
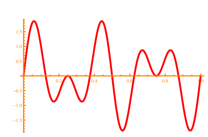
34.2. While the function f contains all the information about the music piece, the computer needs to store this in the form of **data**. The ".WAV" file for example contains 44100 sample readings per second. We can not hear higher than 20'000 KHz. A theorem of Nyquist-Shannon assures that 44.1KHz is good. To get from the sampled values $f(k)$ a function, the **Whittaker-Shannon interpolation formula**

$$f(t) = \sum_{k=1}^{n} f(k) \mathrm{sinc}(t-k)$$

can be used. It involves the **sinc** function $\mathrm{sinc}(x) = \sin(x)/x$.

The wave form and hull

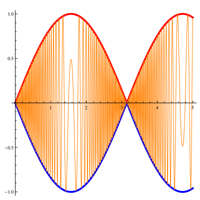
34.3. Periodic signals can serve as **building blocks** of sound. Assume $g(x)$ is a 2π-periodic function, we can generate a sound of 440 Hertz when playing the function $f(x) = g(440 \cdot 2\pi x)$. If the function does not have a smaller period, then we hear the A **tone**. It is a tone with 440 Hertz. We can **modulate** this sound with a **hull function** $h(x)$ and write $f(x) = h(x)g(4402\pi x)$.

34.4. The periodic function g is called a **wave form**. It gives a **timbre** of a sound. Music instruments can be modeled using parameters like **attack**, **vibrato**, **coloration**, **noise**, **echo** or **reverberation**.

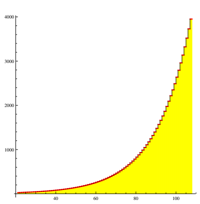
Definition: The **hull function** $h(x)$ is an interpolation of successive local maxima of f.

34.5. For the function $f(x) = \sin(100x)$ for example, the hull function is $h(x) = 1$ For $f(x) = \sin(x)\sin(100x)$ the hull function is $h(x) = |\sin(x)|$. With slight abuse of notation, we sometimes just say $\sin(x)$ is the hull function as the function is sandwiched by the envelopes $\sin(x)$ and $-\sin(x)$.



We can not hear the actual function $f(x)$ because it changes too fast. We can not follow the individual vibrations. But we can hear the hull function. as well as **large scale amplitude** changes like **creshendi** or **diminuendi** or a **vibrato**. When playing frequencies that are close, we can notice **interference**, the sound analogue of **Moiré patterns** in optics.

The scale



34.6. Western music uses a discrete set of frequencies. This scale is based on the **exponential function**. The frequency f is an exponential function of the scale s. On the other hand, if the frequency is known then the scale number is a logarithm. This is a nice application of the logarithm:

Definition: A frequency f has the **Midi number** $s = 69 + 12 \cdot \log_2(f/440)$. The **piano scale function** or **midi function** gives back $f(s) = 440 \cdot 2^{(s-69)/12}$.

34.7. The Midi tone $s = 100$ for example is a sound of $f = 2637.02$ Hertz (oscillations per second).

The **piano scale function** $f(s) = 440 \cdot 2^{(s-69)/12}$ is an exponential function $f(s) = be^{as}$ which satisfies $f(s + 12) = 2f(s)$.

```
midifrequency[m_] := N[440 2^((m - 69)/12)]
```

34.8. A classical piano has 88 keys which scale from 21 to 108. The frequency ranges from $f = 27.5Hz$, the sub-contra-octave A, to the highest $f = 4186.01Hz$, the 5-line octave C.

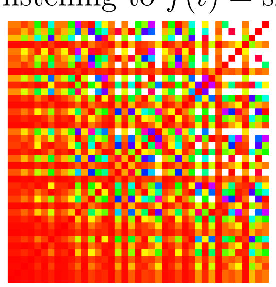
34.9. Filters: a function can be written as a sum of sin and cos functions. Our ear does this so called **Fourier decomposition** automatically. We can so hear melodies, filter out part of the music and hum it.
Pitch and autotune: it is possible to filter out frequencies and adapt their frequency. The popular filter **autotune** moves the frequencies around correcting wrong singing. If 440 Hertz (A) and 523.2 Hertz (C) for example were the only allowed frequencies, the filter would change a function $f(x) = \sin(2\pi 441x) + 4\cos(2\pi 521x)$ to $g(x) = \sin(2\pi 440x) + 4\cos(2\pi 523.2x)$. **Rip and remix**: if f and g are two songs, we can build the average $(f+g)/2$. A composer does this using **tracks**. Different instruments are recorded independently and then mixed together. A guitar $g(t)$, a voice $v(t)$ and a piano $p(t)$ together can form $f(t) = ag(t)+bv(t)+c(p(t)$ with suitably chosen constants a, b, c. **Reverberate and echo**: if f is a song and h is some time interval, we can look at $g(x) = Df(x) = [f(x+h) - f(x)]/h$. For small h, like $h = 1/1000$ the song does not change much because hearing $\sin(kx)$ or $\cos(kx)$ produces the same song. However, for larger h, one can get **reverberate** or **echo** effects.

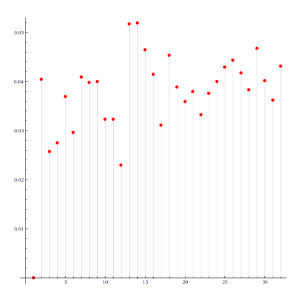
34.10. Mathematics and music have a lot of overlap. Besides wave form analysis and music manipulation operations and symmetry, there are **encoding and compression problems**. A **Diophantine problem** is the question how well a frequency can be approximated by rationals. Why is the **chromatic scale** based on $2^{1/12}$ so effective? **Indian music** for example uses **micro-tones** and a scale of 22. The 12-tone scale has the property that many powers $2^{k/12}$ are close to rational numbers. This can be quantified with the **scale fitness**

$$M(n) = \sum_{k=1}^{n} \min_{\mathrm{p,q}} |2^{k/n} - \frac{p}{q}| G(p/q)$$

where $G(n/m)$ is Euler's **gradus suavitatis** (="degree of sweetness") defined as $G(n/m) = 1 + \sum_{p|n*m}(p-1)$ in which the sum runs over all prime factors p of $n*m$. For example $G(3/4) = 1 + (2-1) + (2-1) + (3-1)$ because $3*4 = 12 = 2*2*3$.

34.11. The figure below illustrates why the 12-tone scale minimizes $M(n)$. We could also replace the concept of octave. Stockhausen experimented with replacing 2 with 5 and used the **Stockhausen scale** $5^{k/25}$. It is $f(t) = \sin(2\pi t 440 \cdot 5^{[t]/25})$, where $[t]$ is the largest integer smaller than t. The familiar **12-tone scale** can be admired by listening to $f(t) = \sin(2\pi t 440 \cdot 2^{[t]/12})$.





The perfect fifth 3/2 has the gradus suavitatis $1 + E(6) = 1 + 2 = 3$ which is the same than the perfect fourth 4/3 for which $1 + E(12) = 1 + (2-1)(3-1)$. You can listen to the perfect fifth $f(x) = \sin(1000x) + \sin(1500x)$ or the perfect fourth $\sin(1000x) + \sin(1333x)$ and here is a function representing an **accord** with four notes $\sin(1000x)+\sin(1333x)+\sin(1500x) + \sin(2000x)$.

Homework

Problem 34.1: Modulation. Draw and play the following function

$$f(x) = \cos(4000x) - \cos(4011x)$$

for three seconds. You can use Wolfram alpha to do that Compare it with when playing

$$f(x) = \cos(4000x) + Cos[2000x]$$

Here is how to play a function with Mathematica:

```
Play[Cos[x] Sin[Exp[2 x]]/x, {x, 0,9}]
```

Problem 34.2: Amplitude modulation (AM): If you listen to $f(x) = |\cos(x^2)| \sin(1000x)$ you hear an amplitude change. Draw the hull function or listen to it and count how many increases in amplitudes to you hear in 10 seconds.

Problem 34.3: Other tonal scales, Midi number: As a creative musician, we create our own tonal scale. You decide to take the 8'th root of 3 as your basic frequency change from one tone to the next.
a) After how many tonal steps has the frequency f tripled?
b) Build the midi function and then write down the inverse for your tonal scale.

Problem 34.4: a) What is the frequency of the Midi number $s = 22$?
b) Which midi number belongs to the frequency $f = 2060 Herz$?

Problem 34.5: Log and Exp rules. a) Give an example showing $a^{(b^c)} \neq (a^b)^c$.
b) Simplify $\ln(e^{100} \cdot e^{50})$.
c) Which of the following expressions are integers? $\ln(e^{\ln(e^{10})})$, $\ln(e+e)$, $\ln(e)+\ln(e)$, $\ln(e) + \ln(e^2)$, $\ln(e^{\ln(e)})$.
d) Which of the following expressions are true $\ln(a + b) = \ln(a) + \ln(b)$, $\ln(ab) = \ln(a) \cdot \ln(b)$, $\ln(ab) = \ln(a) + \ln(b)$, $\ln(a^b) = \ln(a)\ln(b)$? e) Which of the following expressions are true $e^{a+b} = e^a + e^b$, $e^{ab} = e^a e^b$, $e^{a+b} = e^a e^b$, $(e^b)^c = e^{bc}$, $e^{(b^c)} = e^{bc}$?