Unit 30: Partial Fractions

30.1. The method of **partial fractions** allows to extend the range of integrals we can do. It is just about **algebra**. As we know how to integrate polynomials like x^4+5x+3 we would like to be able to integrate rational functions $f(x) = \frac{p(x)}{q(x)}$, where p, q are polynomials. As we know already the integrals of $1/(x+a)$ and $1/(1+x^2)$, why not try to write a general fraction using such terms.

30.2. First of all, we should be reminded on how to calculate with fractions. How do we add, subtract, multiply or divide fractions? What does it mean to take a fraction to a power or the exponential of a fraction?

Definition: The **partial fraction method** writes $p(x)/q(x)$ as a combination of functions of the form $A/(x+a)$, which we can integrate.

30.3.

In order to integrate $\int \frac{1}{(x-a)(x-b)}\, dx$, write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} .$$

cross multiply:

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)} .$$

and match the nominator:

$$1 = Ax - Ab + Bx - Ba .$$

This implies $A+B=0, Ab-Ba=1$ allowing to solve for A, B.

30.4. Example: To integrate $\int \frac{2}{1-x^2}\, dx$ we can write

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

and integrate each term

$$\int \frac{2}{1-x^2} = \ln(1+x) - \ln(1-x) .$$

30.5. Example: Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} .$$

Now multiply out and solve for A, B:

$$A(x-2) + B(x-3) = 5 - 2x .$$

This gives the equations $A + B = -2, -2A - 3B = 5$. From the first equation we get $A = -B - 2$ and from the second equation we get $2B + 4 - 3B = 5$ so that $B = -1$ and so $A = -1$. We have not obtained

$$\frac{5-2x}{x^2-5x+6} = -\frac{1}{x-3} - \frac{1}{x-2}$$

and can integrate:

$$\int \frac{5-2x}{x^2-5x+6}\, dx = -\ln(x-3) - \ln(x-2) .$$

30.6. Example: Integrate $f(x) = \int \frac{1}{1-4x^2}\, dx$. **Solution.** The denominator is factored as $(1-2x)(1+2x)$. Write

$$\frac{A}{1-2x} + \frac{B}{1+2x} = \frac{1}{1-4x^2} .$$

We get $A = 1/4$ and $B = -1/4$ and get the integral

$$\int f(x)\, dx = \frac{1}{4}\ln(1-2x) - \frac{1}{4}\ln(1+2x) + C .$$

30.7. The following **Hospital method** or **residue method** saves time especially with many functions where we would a complicated system of linear equations would have to be solved.

If a is different from b, then the coefficients A, B in

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} ,$$

are

$$A = \lim_{x\to a}(x-a)f(x) = p(a)/(a-b),\ B = \lim_{x\to b}(x-b)f(x) = p(b)/(b-a) .$$

The reason is: if we multiply the identity with $x-a$ we get

$$\frac{p(x)}{(x-b)} = A + \frac{B(x-a)}{x-b} .$$

Now we can take the limit $x \to a$ without peril and end up with $A = p(a)/(x-b)$.

30.8. Example: Find the anti-derivative of $f(x) = \frac{2x+3}{(x-4)(x+8)}$. **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2*4+3}{4+8} = 11/12$, and $B = \frac{2*(-8)+3}{(-8-4)} = 13/12$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8} .$$

The integral is

$$\frac{11}{12}\ln(x-4) + \frac{13}{12}\ln(x+8) .$$

30.9. Example: Find the anti-derivative of $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$. **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$ and $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$ and $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$. The integral is

$$\frac{3}{2}\ln(x-1) - 7\ln(x-2) + \frac{13}{2}\ln(x-3) .$$

30.10. A remark: it is always amazing to see how fast many students can do this, even with many variables. Once you realize the principle, the computation is very nice. Mathematically one has understood the "residue method" which is also important in complex analysis, which is calculus done when the real numbers are extended to complex numbers.

> To get the constant A in the part $A/(x-a)$, just divide out the factor $x-a$ from the original expression for $f(x)$, then set $x = a$.

30.11. Example: Let

$$f(x) = \frac{1}{(x-3)(x-5)(x-11)(x-4)} = A/(x-3)+B/(x-5)+C/(x-11)+D/(x-4) .$$

To get the constant C for example, just delete the factor $(x-11)$ in the left part, then put $x = 11$:

$$C = \frac{1}{(x-3)(x-5)(x-4)} = \frac{1}{(11-3)(11-5)(11-4)} = 1/(8*6*7) .$$

Homework

Problem 30.1: Compute the following integrals by simplification. There is no partial fraction needed, you just should see how to simplify the expression.
a) $\int \frac{x+1}{x}\, dx$.
b) $\int \frac{x+1}{x^2-1}\, dx$.
c) $\int \frac{x^3+3x^2+3x+1}{x^2+2x+1}\, dx$.
d) $\int \frac{x^2-2x-3}{x^2-5x+6}\, dx$.
e) $\int \frac{x-1}{x+1}\, dx$.

Problem 30.2: Solve the following integrals using partial fraction
a) $\int \frac{1}{x^2-4}\, dx$.

b) $\int \frac{1}{(x-1)(x+1)(x-3)}\, dx$.

Problem 30.3: And now these two
a) $\int \frac{1}{x^2-14x+45}\, dx$

b) $\int \frac{2}{x^2-9}\, dx$

Problem 30.4: As this is the last lecture on integration, lets also review other methods. Solve the following integrals:
a) Compute $\int (x-1)^7 \sin(3x)\, dx$.

b) Compute $\int \frac{1}{(1+x^2)(1+\arctan(x)^2)}\, dx$.

Problem 30.5: For this example, you really need to use the Hospital method: $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)}\, dx$. Write the function as

$$f = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+7} + \frac{D}{x-3}$$

then figure out the constants A, B, C, D.

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