INTRODUCTION TO CALCULUS

MATH 1A

Unit 29: Integration by parts

29.1. Integration by parts is based on the **product rule** $(uv)' = u'v + uv'$. It complements the method of substitution we have seen last time and which had been reversing the **chain rule**. As a rule of thumb, always try first to **1) simplify a function and to integrate using known functions**, then **2) try substitution** and finally **3) try integration by parts**.

$$\int \text{u(x)}\ \text{v'}\,(x)dx = u(x)v(x) - \int u'(x)v(x)\ dx.$$

29.2. Lets try this with $\int x\sin(x)\ dx$. First identify what you want to differentiate and call it u, the part to integrate is called v'. Now, write down uv and subtract a new integral which integrates $u'v$:

$$\int \text{x}\ \sin(x)\ dx = \text{x}\ (-\cos(x)) - \int 1\ (-\cos(x))\ dx = -x\cos(x) + \sin(x) + C\ dx\ .$$

On paper, we can stream-line this by just placing an **down-arrow** under the expression you differentiate and an **up-arrow** under the expression you integrate. You remember to first integrate, then subtract the integral of the expression where you both integrate and differentiate. If you like to write down the u, v's, do so. What you need to remember is $\boxed{\int udv = uv - \int vdu}$.

29.3. Find $\int xe^x\ dx$. **Solution.** You want to differentiate x and integrate e^x.

$$\int \text{x}\ \exp(x)\ dx = x\exp(x) - \int 1\cdot\exp(x)\ dx = x\exp(x) - \exp(x) + C\ dx\ .$$

29.4. Find $\int \ln(x)\ dx$. **Solution.** While there is only one function here, we need two to use the method. Let us look at $\ln(x)\cdot 1$:

$$\int \ln(x)\ 1\ dx = \ln(x)x - \int \frac{1}{x}x\ dx = x\ln(x) - x + C\ .$$

29.5. Find $\int x\ln(x)\ dx$. **Solution.** Since we know from the previous problem how to integrate ln we could proceed by taking $x = u$. We can also take $u = \ln(x)$ and $dv = x$:

$$\int \ln(x)\ \text{x}\ dx = \ln(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\ dx$$

which is $\ln(x)x^2/2 - x^2/4$.

29.6. We saw that it is good to differentiate ln's. The word LIATE (explained below) tells which functions we want to call u and differentiate.

Marry go round: Find $I = \int \sin(x)\exp(x)\ dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x)\exp(x) - \int \cos(x)\exp(x)\ dx\ .$$

Lets do it again:

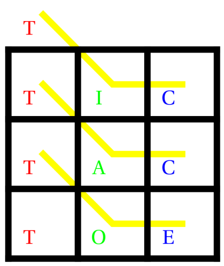
$$= \sin(x)\exp(x) - \cos(x)\exp(x) - \int \sin(x)\exp(x)\ dx.$$

We moved in circles and are stuck! But not really! We have derived an identity

$$I = \sin(x)\exp(x) - \cos(x)\exp(x) - I$$

which we can solve for I and get $\boxed{I = [\sin(x)\exp(x) - \cos(x)\exp(x)]/2.}$

Tic-Tac-Toe



Integration by parts can become complicated if we need to repeat it several times. Keeping the order of the signs can be especially daunting. Fortunately, there is a powerful **tabular integration by parts method**. It has been called **"Tic-Tac-Toe"** in the movie Stand and deliver. Lets call it **Tic-Tac-Toe** therefore.

29.7. Find the anti-derivative of $(x-1)^3 e^{2x}$. **Solution:**

$(x-1)^3$	$\exp(2x)$	
$3(x-1)^2$	$\exp(2x)/2$	$\oplus$
$6(x-1)$	$\exp(2x)/4$	$\ominus$
6	$\exp(2x)/8$	$\oplus$
0	$\exp(2x)/16$	$\ominus$

The anti-derivative is

$$(x-1)^3 e^{2x}/2 - 3(x-1)^2 e^{2x}/4 + 6(x-1)e^{2x}/8 - 6e^{2x}/16 + C\ .$$

29.8. Find the anti-derivative of $x^2\cos(x)$. **Solution:**

x^2	$\cos(x)$	
$2x$	$\sin(x)$	$\oplus$
2	$-\cos(x)$	$\ominus$
0	$-\sin(x)$	$\oplus$

The anti-derivative is $x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$.

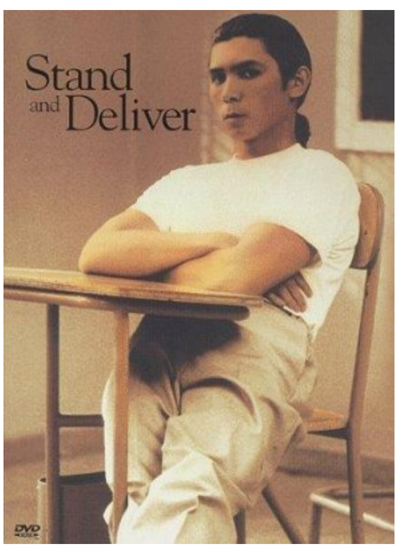
29.9. Find the anti-derivative of $x^7 \cos(x)$. **Solution:**

x^7	$\cos(x)$	
$7x^6$	$\sin(x)$	$\oplus$
$42x^5$	$-\cos(x)$	$\ominus$
$120x^4$	$-\sin(x)$	$\oplus$
$840x^3$	$\cos(x)$	$\ominus$
$2520x^2$	$\sin(x)$	$\oplus$
$5040x$	$-\cos(x)$	$\ominus$
5040	$-\sin(x)$	$\oplus$
0	$\cos(x)$	$\ominus$

The anti-derivative is

$$\begin{aligned} F(x) &= x^7\sin(x) + 7x^6\cos(x) - 42x^5\sin(x) \\ &- 210x^4\cos(x) + 840x^3\sin(x) + 2520x^2\cos(x) \\ &- 5040x\sin(x) - 5040\cos(x) + C \ . \end{aligned}$$

29.10. Do this without this method and you see the value of the method.
[1] [2] [3]:



I myself learned the method from the movie "Stand and Deliver", where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to the article of V.N. Murty.

$$\begin{aligned} \int fg\,dx &= fg^{(-1)} - f^{(1)}g^{(-2)} + f^{(2)}g^{(-3)} - \dots \\ &- (-1)^n \int f^{(n+1)}g^{(-n-1)}\,dx \end{aligned}$$

The Tic-Tac-Toe method can be verified by induction because the f function is differentiated again and again and the g function is integrated again and again. The alternating minus-plus-signs come from the fact that we subtract an integral and so change the sign of both. We always pair a k'th derivative with a $k+1$'th integral and take the sign $(-1)^k$.

Coffee or Tea?

[1]V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, p. 90-94.
[2]D. Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, p. 307-311.
[3]K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, p. 542-543

29.11. We want to first differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to "lipton" (the tea).

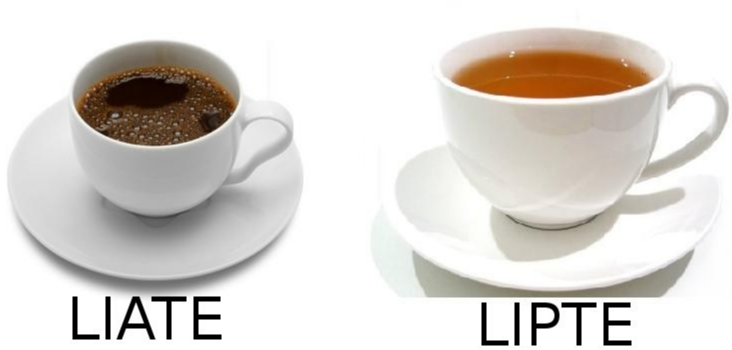
For coffee lovers, there is **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials. Now, **LIATE** is close to "latte" (the coffee).

Whether you prefer to remember it as a **"coffee latte"** or a **"lipton tea"** is up to you.

If you should not like neither coffee, nor tea, there is "opportunist method":

Just integrate what you can integrate and differentiate the rest.

An don't forget to consider integrating 1, if nothing else works.



Homework

Problem 29.1: Integrate $\int x^3 \ln(x)\, dx$.

Solution:
Differentiate ln and integrate x^3.

$$\int x^3 \ln(x)\, dx = x^4 \ln(x)/4 - \int x^4/4(1/x)\, dx = x^4 \ln(x)/4 - x^4/16 + C\ .$$

This was similar than what we have done in class.

Problem 29.2: Integrate $\int x^5 \sin(x)\, dx$

Solution:
It is best here to use tabular integration.

x^5	$\sin(x)$	
$5x^4$	$-\cos(x)$	$\oplus$
$20x^3$	$-\sin(x)$	$\ominus$
$60x^2$	$\cos(x)$	$\oplus$
$120x^1$	$\sin(x)$	$\ominus$
120	$-\cos(x)$	$\oplus$

The anti-derivative is

$-x^5\cos(x) + 5x^4\sin(x) + 20x^3\cos(x) - 60x^2\sin(x) - 120x\cos(x) + 120\sin(x) + C$.

Problem 29.3: Find the anti derivative of $\int 2x^6\exp(x)\,dx$.

Solution:
Also here, tabular integration rules.

$2x^6$	$\exp(x)$	
$12x^5$	$\exp(x)$	$\oplus$
$60x^4$	$\exp(x)$	$\ominus$
$240x^3$	$\exp(x)$	$\oplus$
$720x^2$	$\exp(x)$	$\ominus$
$1440x^1$	$\exp(x)$	$\oplus$
1440	$\exp(x)$	$\ominus$

The anti derivative is $e^x(2x^6 - 12x^5 + 60x^4 - 240x^3 + 720x^2 - 1440x + 1440) + C$.

Problem 29.4: Find the anti derivative of $\int \sqrt{x}\ln(x)\,dx$.

Solution:
Differentiate ln and integrate $\sqrt{x}$

$$\ln(x)x^{3/2}(2/3) - \int x^{3/2}(2/3)1/x\,dx = \ln(x)x^{3/2}(2/3) - (4/9)x^{3/2} .$$

Problem 29.5: Find the anti derivative of $\int \sin(x)\exp(-x)\,dx$.

Solution:
This is a **Merry-Go-Round problem**. Differentiate $\exp(x)$ and integrate $\sin(x)$. Call the integral I.

$$I = -\cos(x)\exp(-x) - \int \cos(x)\exp(-x)\,dx$$

Now differentiate again $\exp(x)$ and integrate $\cos(x)$:

$$-\cos(x)\exp(-x) + \sin(x)\exp(-x) - \int \sin(x)\exp(-x)\,dx$$

We ended up with the same integral I on the right hand side. Solve the equation

$$I = -\cos(x)\exp(-x) - \sin(x)\exp(-x) - I$$

for I to get

$$I = [-\cos(x)\exp(-x) - \sin(x)\exp(-x)]/2 .$$

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