INTRODUCTION TO CALCULUS

MATH 1A

Unit 28: Substitution

28.1. Functions like e^{6x} or $1/(1+x)$ have been integrated by seeing $\boxed{\int f(cx+a)\, dx = F(cx+a)/c}$, if $F' = f$. We have also seen how to "spot the chain rule": if $f(x) = g(u(x))u'(x)$, $\boxed{\int f(x)\, dx = G(u(x)) + C}$, where $G' = g$.

28.2. Example: $\int e^{x^4+x^2}(4x^3+2x)\, dx = e^{x^4+x^2} + C$.
Example: $\int \sqrt{x^5+1}x^4\, dx = (2/15)(x^5+1)^{3/2}$.
Example: $\int \frac{\ln(x)}{x}\, dx = \ln(x)^2/2 + C$

28.3. The **method of substitution** formalizes this and makes it more systematic: **A)** select part of the formula, call it u. **B)** write $du = u'dx$ and **C)** replace dx with du/u'. **D)** If all terms x have disappeared, integrate. **E)** Back substitute the variable x. If things should not work, go back to A) and try an other u.

$$\int f(\text{u(x)})\, \text{u'(x)} \quad \text{dx} = \int g(\text{u})\, \text{du}\ .$$

28.4. Example: to get $\int \ln(x)/x\, dx$, pick $u = \ln(x)$, compute $du = (1/x)dx$ and so $dx = xdu$. We get $\int udu = u^2/2 + C$. Back substitution gives $\ln^2(x)/2 + C$.
Example: to get $\int \ln(\ln(x))\frac{1}{\ln(x)x}\, dx$, try $u = \ln(x)$ and $du = (1/x)dx$, then plug this in to get $\int \ln(u)/u\, du = \ln^2(u)/2 + C$. Back substitute to get $\ln^2(\ln(x)) + C$.

Example: To get $\int \frac{x}{1+x^4}\, dx$, substitute $u = x^2, du = 2xdx$ to get gives $(1/2)\int du/(1+u^2)\, du = (1/2)\arctan(u) = (1/2)\arctan(x^2) + C$.

Example: To get $\sin(\sqrt{x})/\sqrt{x}$, try $u = \sqrt{x}, x = u^2, dx = 2udu$. The result is $-2\cos(\sqrt{(x)}) + C$.

28.5. Example: (harder) $\int \frac{x^3}{\sqrt{x^2+1}}\, dx$ might trigger the reflex $u = \sqrt{x^2+1}$ but this does not work does not work. With $u = x^2+1$ and $du = 2xdx$ and $dx = du/(2\sqrt{u-1})$. We get

$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}}\, du = \int \frac{(u-1)}{2\sqrt{u}} = \int \frac{u^{1/2}}{2} - \frac{u^{-1/2}}{2}\, du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2}\ .$$

28.6. For **definite integrals** $\int_a^b f(x)\,dx$, one could find an anti-derivative as described and fill in the bounds.

$$\int_a^b g(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} g(u)\,du\ .$$

Proof. The right hand side is $G(u(b))-G(u(a))$ by the fundamental theorem of calculus. The integrand on the left has an anti derivative $G(u(x))$. Again by the fundamental theorem of calculus the integral leads to $G(u(b))-G(u(a))$.

Example: $\int_0^1 \frac{1}{5x+1}\,dx = [\ln(u)]/5|_1^6 = \ln(6)/5$.
Example: $\int_3^5 \exp(4x-10)\,dx = [\exp(10)-\exp(2)]/4$.

Homework

Problem 28.1: Find the following anti-derivatives. But do it formally following the steps, even if you see the answer already.
a) $\int x^2\sin(x^3)\,dx$
b) $\int e^{x^6+x}(6x^5+1)\,dx$
c) $\int -\cos(\sin(3x))\cos(3x)/3\,dx$
d) $\int e^{\tan(2x)}/\cos^2(2x)\,dx$.

Solution:
a) $-\cos(x^3)/3+C$.
b) $e^{x^6-x}+C$.
c) $\sin(\sin(3x))$.
d) $e^{\tan(2x)}/2+C$.

Problem 28.2: Compute the following definite integrals. It is fine to find first an anti-derivative and only at the end place the bounds:
a) $\int_1^2 \sqrt{x^5+x}(5x^4+1)\,dx$
b) $\int_0^{\sqrt{\pi}} 6\sin(x^2)x\,dx$.
c) $\int_e^{e^2} \frac{\sqrt{\ln(x)}}{x}\,dx$
d) $\int_0^1 \frac{5x}{\sqrt{1+x^2}}\,dx$.

Solution:
a) Substitute $u = x^5 + x$. The anti derivative is $(2/3)(x + x^5)^{3/2}/5$. Evaluating gives 130.283. b) The anti-derivative is $-3\cos(x^2)$. The definite integral is 6.
c) The anti-derivative is $2(\ln(x)^{3/2})/3$. The definite integral is 1.2189.
d) The anti-derivative is $5\sqrt{1+x^2}$. The numerical answer is 2.071.

Problem 28.3: Compute $\int_e^{2e} \frac{dx}{\sqrt{\ln(x)x}}$.

Solution:
Substitute $u = \ln(x)$. The anti-derivative is $2\sqrt{\ln(x)}$. Numerically we get 0.60242.

Problem 28.4: Find the indefinite integrals:
a) $\int \frac{x^5}{\sqrt{x^2+1}}\, dx$.

b) $\int \frac{1}{x(1+\ln(x)^2)}\, dx$

Solution:
a) Substitute $u = 1+x^2$, then fill in $x = \sqrt{u-1}$. The expression is a bit more complicated. We end up with $\sqrt{1+x^2}(3x^4 - 4x^2 + 8)/15$.
b) Substitute $u = \ln(x)$. We get $\arctan(\ln(x)) + C$.

Problem 28.5: Find the anti-derivatives of a) $\frac{\cos(x^3)}{e^{\sin(x^3)}} x^2$

b) $\cot(\sqrt{x})/\sqrt{x}$.

Solution:
Substitute $u = \sin(x^3)$. This gives $\frac{1}{3}e^{-\sin\left(x^3\right)}$. b) $2\ln(\sin(\sqrt(x))$.

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