INTRODUCTION TO CALCULUS

MATH 1A

Unit 27: Sigmoid function

27.1. What is $\frac{d}{dx}\int_0^x f(t)\,dt$? If F is an anti-derivative, this is $\frac{d}{dx}[F(x)-F(0)] = f(x)$. [1] There are two aspects of the **fundamental theorem**:

$$\int_0^x f'(t)\,dt = f(x) - f(0), \qquad \frac{d}{dx}\int_0^x f(t)\,dt = f(x)\,.$$

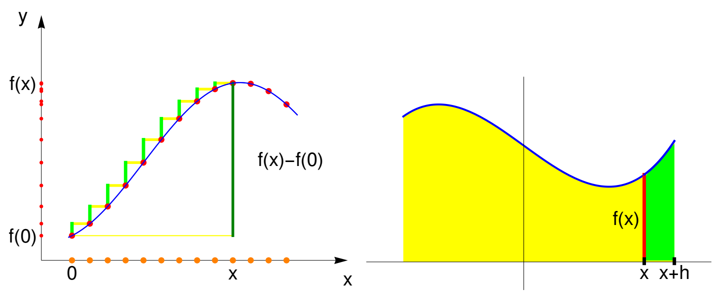


FIGURE 1. Integrate the derivative or differentiate the integral.

27.2. The **activation function** for **neural networks** is given by a differentiable function like $\sigma(x) = (\tanh(x/2)+1)/2 = e^x/(1+e^x)$ rather than a **step function** $(\text{sign}(x)+1)/2$. The first one is the **sigmoid function**. You work on this a bit in this homework.

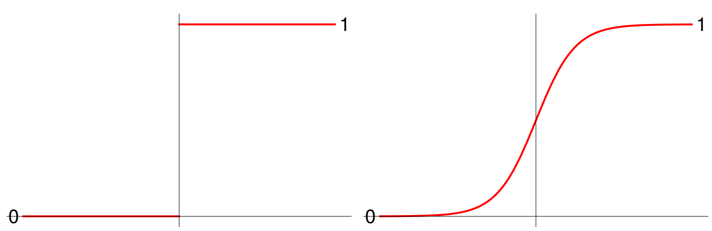


FIGURE 2. The **step function** $(\text{sign}(x)+1)/2$ is non-differentiable, the **sigmoid function** $(\tanh(x/2)+1)/2 = e^x/(1+e^x)$ is differentiable.

The reason is that differentiability allows to use gradient descent minimum algorithms (GDM) similarly as the Newton method we have seen to find maxima or minima. Sometimes one sees $\sigma(x) = \frac{1}{1+e^{-x}}$. Why is this the same?

[1] All except one student got this wrong in the exam.

Homework: due April 8, 2024

Problem 27.1: Verify $\sigma(x) = \frac{\tanh(x/2)+1}{2}$, where $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ and $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x+e^{-x}}{2}$.

Solution:
$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Now multiply both with e^x to get $\frac{e^{2x}-1}{e^{2x}+1}$ Now add 1 to get $2e^{2x}/(e^{2x}+1) = 2\sigma(2x)$. Having seen $\tanh(x)+1 = 2\sigma(2x)$ is equivalent to $\sigma(x) = [\tanh(x/2)+1]/2$.

Problem 27.2: In this problem we work on the **logistic distribution** in statistics.
a) Check that $F(x) = (\tanh(\frac{x}{2}) + 1)/2$ (which by 27.1) is $\sigma(x)$) has the derivative

$$f(x) = \frac{1}{4\cosh^2(\frac{x}{2})} .$$

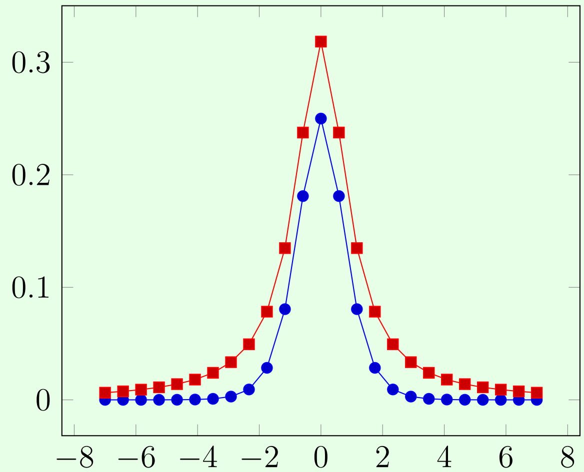
It is called the **logistic distribution**.
b) Why is $\int_{-\infty}^{\infty} f(x)\,dx = 1$? Hint. $\int_{-a}^{a} f(x)\,dx = F(a) - F(-a) = 2\tanh(x/2)$.

Solution:
a) First note that $\sinh' = \cosh$ and $\cosh' = \sinh$ and $\tanh'(x) = 1/\cosh^2(x)$. So $\tan'(x/2)/2 = 1/(4\cosh^2(x/2))$.

Problem 27.3: The function $G(x) = \frac{\arctan(x)}{\pi} + \frac{1}{2}$ resembles $F(x) = \sigma(x)$. Plot both $f = F'$ and $g = G'$ (2 points) then complete the following table (2 points each):

$\frac{d}{dx}\int_0^x \frac{1}{4\cosh^2(\frac{t}{2})}\,dt =$ ☐	$\int_0^x \frac{1}{4\cosh^2(\frac{t}{2})}\,dt =$ ☐
$\frac{d}{dx}\int_0^x \frac{1}{\pi(1+t^2)}\,dt =$ ☐	$\int_0^x \frac{1}{\pi(1+t^2)}\,dt =$ ☐

Solution:



b)

$$\frac{d}{dx}\int_0^x \frac{1}{4\cosh^2(\frac{t}{2})}\,dt = \frac{1}{4\cosh^2(\frac{x}{2})}$$

$$\int_0^x \frac{1}{4\cosh^2(\frac{t}{2})}\,dt = \tanh(x/2)/2$$

$$\frac{d}{dx}\int_0^x \frac{1}{\pi(1+t^2)}\,dt = \frac{1}{\pi(1+x^2)}$$

$$\int_0^x \frac{1}{\pi(1+t^2)}\,dt = \arctan(x)/\pi$$

Problem 27.4: The sigmoid function $F(x)$ is also called the **standard logistic function** because it satisfies the **logistic equation** $F'(x) = F(x)(1 - F(x))$. Verify this. (Compute both $F'(x)$ and $F(x)(1 - F(x))$ and compare).

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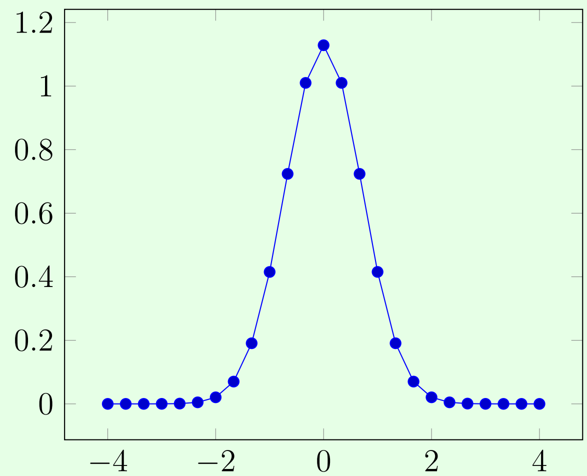
Solution:
Best done with the trig function, where $F'(x) = 1/(4\cosh^2(x/2))$ is already known. Now $4F(x)(1-F(x)) = (\tanh(x/2)-1/2)(\tanh(x/2)+1/2) = \tanh^2(x/2)-1 = 1/\cosh^2(x/2)$.

[2]This is extremely important in machine learning as the derivative is given in terms of the same function. One uses this in backpropagation.

Problem 27.5: The function $N(x) = (1 + \text{erf}(x))/2$ looks similar to the sigmoid function. The error function erf satisfies $\text{erf}'(x) = 2e^{-x^2}/\sqrt{\pi}$.
a) Plot $n(x) = \text{erf}'(x)$.
b) Which of the $f(x), g(x), n(x)$ is the tallest at $x = 0$?
c) Which of the $f(x), g(x), n(x)$ is the tallest at $x = 10$?

Solution:



b) At $x = 0$, the Gaussian is the largest $f(0) = 1/4 = 0.25$
$g(0) = 1/\pi = 0.3183$
$h(0) = 2/\sqrt{\pi} = 1.12$.
and $f(10) = 0.00004$, $g(10) = 0.00315$, $h(10) = 4.18 \cdot 10^{-44}$. Now the Gaussian is the smallest and the Cauchy is the largest.

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