## INTRODUCTION TO CALCULUS

MATH 1A

# Unit 25: Anti derivatives

**25.1.** The following statement follows from the fundamental theorem: take $a = 0, b = x$ and differentiate $\int_0^x f'(t)\, dt = f(x) - f(0)$ with respect to $x$. We get $\frac{d}{dx} \int_0^x f'(t)\, dt = f'(x)$. Replacing $f'$ with $f$ tells us:

$$\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$$

Here is a geometric picture: pick a small $h$ and $y \in [x, x+h]$ so that $f(y)h$ is $\int_x^{x+h} f(t) dt$

$$[\int_0^{x+h} f(t)\, dt - \int_0^x f(t)\, dt]\frac{1}{h} = \int_x^{x+h} f(t) dt \frac{1}{h} = f(y)h\frac{1}{h} = f(y)\ .$$

Taking the limit $h \to 0$ gives the statement of the fundamental theorem. [1]

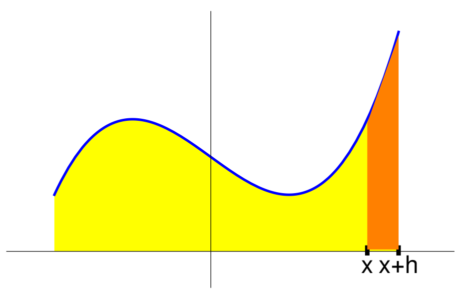



FIGURE 1. The fundamental theorem.

**25.2.** A function $F$ with the property that $F'(x) = f(x)$ is called an **anti-derivative** of $f$. It is not unique but $F(x) = \int_0^x f(t)\, dt$ gives one anti-derivative. Define $\int_b^a f(x) dx = -\int_a^b f(x)\, dx$. This allows to define $F(x)$ for all $x$, also $x < 0$. Here is some more notation: we write an **indefinite integral** with a bare integral sign:

$$\int f(x)\ ; dx = F(x) + C\ ,$$

[1] This argument could be modified to see that the Riemann integral and the fundamental theorem works even for continuous functions.

if $F(x)$ is an anti-derivative. The reason for adding the constant $C$ is that the anti-derivative is not unique. We would write for example $\int \sin(x)dx = -\cos(x) + C$. Note that the constant will disappear if we compute a **definite integral**

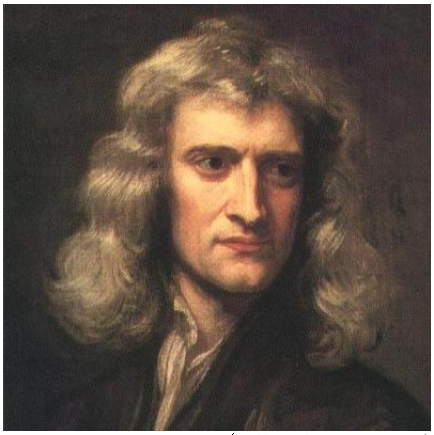
$$\int_a^b f(x) = F(b) - F(a)$$

if $F$ is any anti-derivative. The **indefinite integral** is made up of some anti-derivative plus a constant.

**25.3.** You should have as many **anti-derivatives** "hard wired" in your brain. It really helps. Here are the core functions you should know.

| function | anti derivative |
|---|---|
| $1$ | $x$ |
| $x^n$ | $\frac{x^{n+1}}{n+1}$ |
| $\sqrt{x}$ | $\frac{x^{3/2}2}{3}$ |
| $e^{cx}$ | $\frac{e^{cx}}{c}$ |
| $\cos(cx)$ | $\frac{\sin(cx)}{c}$ |
| $\sin(cx)$ | $\frac{-\cos(cx)}{c}$ |
| $\tan(cx)$ | $\ln(\cos(cx))/c$ |
| $\frac{1}{x}$ | $\ln(x)$ |
| $\frac{1}{1+x^2}$ | $\arctan(x)$ |
| $\ln(x)$ | $x\ln(x) - x$ |

**25.4.** Make your own table!




Newton ("Sir Slightly Annoyed") and Leibniz ("Mr Sour Face")

**Isaac Newton** and **Gottfried Leibniz** are the main figures for the fundamental theorem of calculus.

# Homework: Due Apr 1/2024

**Problem 25.1:** Solve the following indefinite integrals:
a) $\int \sin(\sin(x))\cos(x)\ \ dx$.
b) $\int \csc(x)\sec(x)dx$.
(rewrite $\csc(x) = 1/\sin(x)$, $\sec(x) = 1/\cos(x)$ so that the integrand is $\int \tan(x)\frac{1}{\cos^2(x)}\ dx$).

**Solution:**
a) $\sin(\sin(x)) + C$.
b) $\ln(\tan(x)) + C$.

**Problem 25.2:** Find anti derivatives of the following functions:
a) $1/(1+x)$
b) $1/(1+x^2)$
c) $x/(1+x^2)$
d) $x^2/(1+x^2)$ (will appear in class today)
e) $x^4/(1+x^2)$ (will appear in class today)

**Solution:**
a) $\ln(1+x) + C$.
b) $\arctan(x) + C$.
c) $\ln(1+x^2)\frac{1}{2}$
d) Rewrite as $(x^2+1)/(1+x^2) - 1/(1+x^2)$. The anti-derivative is $x - \arctan(x) + C$.
e) Rewrite as $(x^4-1)/(1+x^2) + 1/(1+x^2) = (x^2-1) + 1/(1+x^2)$. The anti-derivative is $x^3/3 - x + \arctan(x) + C$.

**Problem 25.3:** Evaluate the following indefinite integrals:
a) $\int 3^x + x^3\ dx$
b) $\int 3^{3^x} 3^x\ dx$

**Solution:**
a) Rewrite $e^x = e^{x\ln(3)}$. The answer is $e^{x\ln(3)}/\ln(3) + x^4/4$.
b) From a) we have seen $\int 3^x \, dx = 3^x/\ln(3)$. So that the anti-derivative is $3^{3^x}/\ln(3)^2 + C$.

**Problem 25.4:** See whether you can write down the anti-derivatives in 5 minutes:
a) $\ln(x)/x$
b) $1/\sin^2(3x)$
c) $\frac{1}{\cos(\ln(x))^2}\frac{1}{x}$
d) $\tan(1+2x)$
e) $\frac{\cos(x)}{2\sqrt{\sin(x)}}$

**Solution:**
a) $\ln(x)^2/2 + C$
b) $-\cot(3x)/3 + C$
c) $\tan(\ln(x) + C$
d) $-\ln(\cos(1+2x))/2 + C$
e) $\sqrt(\sin(x)) + C$.

**Problem 25.5:** a) **A clever integral:** Evaluate the following integral (just by being clever, there is no algebra, and no work is needed):

$$\int_{-\pi}^{\pi} \sin(\sin(\sin(\sin(\sin(x))))) \, dx \, .$$

b) **An evil integral**: Evaluate $\int_{e^e}^{e^{e^e}} \frac{1}{\ln(\ln(x))\ln(x)x} \, dx$.

**Solution:**
a) 0 because the function is odd. The area under the curve matches the area above the curve.
b) The anti derivative is $\ln(\ln(\ln(x))) + C$. Evaluating gives $\ln(\ln(\ln(x)))|_{e^e}^{e^{e^e}} = 1 - 0 = 1$.