INTRODUCTION TO CALCULUS

MATH 1A

Unit 24: Fundamental theorem

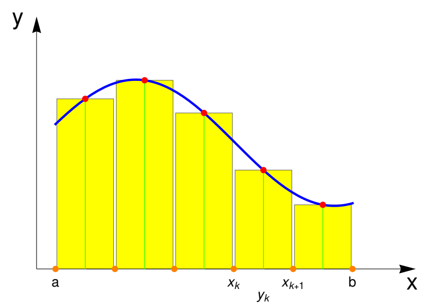
24.1. The **fundamental theorem of calculus** for differentiable functions allows to compute many integrals nicely without having to evaluate nasty and messy sums. First note that in the Riemann sum, we can chose any $y_k \in [x_k, x_{k+1}]$ to get the limit. The case $y_k = x_k$ is called the **left Riemann sum**, the case $y_k = x_{k+1}$ is called the **right Riemann sum**. The following result is also called the "evaluation part of the "fundamental theorem". It is the version is used most:

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Proof. By the **mean value theorem**, there exists in every interval $[x_k, x_{k+1}]$ a number y_k such that $f'(y_k) = (f(x_{k+1}) - f(x_k))/\Delta x$, because $\Delta x = x_{k+1} - x_k$. Now

$$\sum_{k=0}^{n-1} f'(y_k)\Delta x = \sum_{k=0}^{n-1} f(x_{k+1}) - f(x_k) = f(b) - f(a)\ .$$

Taking the limit $n \to \infty$ gives immediately $\int_a^b f'(x)\, dx = f(b) - f(a)$.



24.2. Why is this theorem so nice? One reason is that computing Riemann sums can be difficult. When trying to compute the area under the parabola, for example, we needed to sum up a difficult sum (as we have seen last time in class):

$$\int_0^1 x^2\, dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} (\frac{k}{n})^2 \frac{1}{n} = \lim_{n\to\infty} \frac{(2n-1)(n-1)}{6n^2} = \frac{1}{3}\ .$$

But that is insane. We needed to look up a formula for the sum of the first n squares. For $\int_0^1 x^4\,dx$ we would have to sniff out the formula $\lim_{n\to\infty} \frac{(n-1)(2n-1)\left(3n^2-3n-1\right)}{30n^5}$. We don't want to to do that! [1]

24.3. How do we use this theorem? Given an integral of a function, we ask our-self whether we know a function f such that its derivative is the function we have. We then write $\boxed{\int_a^b f'(x)\,dx = f(x)|_a^b = f(b) - f(a)}$. Notice the intermediate step where we write down the function $f(x)$ and the same bounds from the integral along a vertical bar. For now, we just stare at f' and see whether we can find a function f which has f' as its derivative. Knowing some derivatives is key.

$\int_1^2 x^2\,dx$. We know that x^2 is the derivative of $x^3/3$. So $\int_1^2 x^2\,dx = \frac{x^3}{3}|_1^2 = 7/3$.

$\int_0^\pi \sin(x)\,dx$. We know that $\sin(x)$ is the derivative of $-\cos(x)$. So $\int_0^\pi \sin(x)\,dx = -\cos(x)|_0^\pi = 2$.

$\int_0^5 x^7\,dx = \frac{x^8}{8}|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

$\int_0^{\pi/2} \cos(x)\,dx = \sin(x)|_0^{\pi/2} = 1$.

Find $\int_0^\pi \sin(x)\,dx$. **Solution**: The answer is 2.

For $\int_0^2 \cos(t+1)\,dt = \sin(x+1)|_0^2 = \sin(2) - \sin(1)$, the additional term $+1$ does not make matter as when using the chain rule, it goes away.

For $\int_{\pi/6}^{\pi/4} \cot(x)\,dx$, the anti-derivative is difficult to spot. It becomes only accessible if we know, where to look: the function $\ln(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\ln(\sin(x))|_{\pi/6}^{\pi/4} = \ln(\sin(\pi/4)) - \ln(\sin(\pi/6)) = \ln(1/\sqrt{2}) - \ln(1/2) = -\ln(2)/2 + \ln(2) = \ln(2)/2$.

[1] You could although like with $\mathrm{Sum}[(k/n)^4/n, \{k, 0, n-1\}]/n$ or by asking any AI. Such formulas are well known in the literature and so no problem for any AI. They don't feel any pain when they are asked to do it for x^{100}. Or do they? Maybe they are just still too polite. In 20 years, they might punish you with a few days of misinformation and there is nothing you can do about it!

Homework: Due Mar 29/2024

Problem 24.1: Find a function f such that f' is what you see inside, then integrate
a) $\int_{-1}^{1} 4x^3 + 30x^2 \, dx$.
b) $\int_0^1 (x+1)^5 \, dx$.

Solution:
a) $x^4 + 30x^3/3|_{-1}^{1} = 20$,
b) $(x+1)^6/6|_0^1 = 21/2$.

Problem 24.2: Evaluate the following integrals:
a) $\int_2^3 5/(x-1) \, dx$,
b) $\int_0^{\sqrt{\pi}} \sin(x^2) 4x \, dx$

Solution:
a) $5\ln(x-1)|_2^3 = \ln(32)$,
b) $-2\cos(x^2)|_0^{\sqrt{\pi}} = 4$.

Problem 24.3: Evaluate the following integrals:
a) $\int_1^2 2^x \, dx$,
b) $\int_0^{\sqrt{3}} \frac{1}{1+x^2} \, dx$,

Solution:
a) First write $2^x = e^{x\ln(2)}$, then integrate: $2/\ln(2)$
b) We have $\arctan x|_0^{\sqrt{3}} = \pi/3$.

Problem 24.4: Also here, in each case, just guess what derivative the integrand is.
a) $\int_0^{\sqrt(\pi)} \sin(x^2)x\ dx$
b) $\int_0^3 (3/2)\sqrt{1+x}\ dx$
c) $\int_0^{\sqrt{\ln(2)}} 4xe^{-x^2}\ dx$
d) $\int_e^{e^2} \frac{5}{x\ln(x)}\ dx$ e) $\int_0^1 \cos\sin\sin(x)\cos(\sin(x))\cos(x)\ dx$

Solution:
a) 1.
b) 7.
c) 1/4.
d) $5\ln(2)$.
e) $\sin\sin\sin(1)$

Problem 24.5: In this problem, we look at a situation where x appears in the bound.
a) Compute $F(x) = \int_0^{x^3} \sin(t)\ dt$, then find $F'(x)$.
b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t)\ dt$ then find $G'(x)$

About the notation in 24.5 Note that writing $\int_0^{x^3} \sin(x)\ dx$ would have been **ill defined**. Computer scientists call such things "notation overload". It is commonly done, but confusing. Many programming languages allow to do that, but making use of it is a common source for programming errors. Mathematica for example allows to do it. The expression Integrate$[Sin[x], \{x, 0, x^3\}]$ gives the right answer. It internally rewrites this as Integrate$[Sin[t], \{t, 0, x^3\}]$. But the AI is well aware that it must have been a greenhorn who has entered the input. It is polite and does not tell you what it thinks about you. It might add you to an internal "fools" database. And again, there is nothing you can do about it.

Solution:
a) $F(x) = -\cos(x^3) + 1$ and $F'(x) = 3\sin(x^3)x^2$.
b) $G(x) = e^{\cos(x)} - e^{\sin(x)}$ and $G'(x) = -\sin(x)e^{\cos(x)} - \cos(x)e^{\sin(x)}$.