

# INTRODUCTION TO CALCULUS

MATH 1A

## Unit 23: Riemann integrals

**23.1.** In this lecture, we define the **definite integral**  $\int_a^b f(t) dt$  if  $f$  is a differentiable function. It has an interpretation as an **area under the curve**. Define  $x_k = a + k\Delta x$  where  $k = 0, \dots, n-1$  and  $\Delta x = (b-a)/n$ . The sum

$$S_n f = [f(x_0) + \dots + f(x_{n-1})]\Delta x$$

is called a **Riemann sum**. It is a sum of areas of small rectangles of width  $\Delta x$  and height  $f(x_k)$ . It is a “left Riemann sum” because we evaluate the function to the left of the intervals.

**Definition:** Define

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} f(x_k) \Delta x .$$



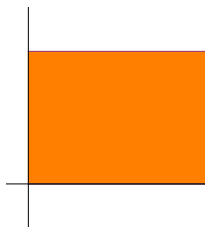
**23.2.** A very important result is that

For any differentiable function, the limit exists.

We can explicitly estimate the error: there are  $n$  little pieces where the region differs from the rectangle union. Each of these pieces has area  $\leq M/n$ , where  $M$  is the maximal slope that  $f$  can have in the given interval.

For non-negative  $f$ , the value  $\int_0^x f(x) dx$  is the **area between the x-axis and the graph** of  $f$ . For general  $f$ , it is a **signed area**, the difference between two areas.

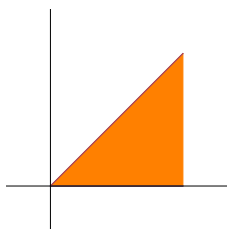
If  $f(x) = c$  is constant, then  $\int_0^x f(t) dt = cx$ .



Let  $f(x) = cx$ . The area is half of a rectangle of width  $x$  and height  $cx$  so that the area is  $cx^2/2$ . Adding up the Riemann sum is more difficult. Let  $k$  be the largest integer smaller than  $xn = x/h$ . Then

$$S_n f(x) = \frac{1}{n} \sum_{j=1}^k \frac{cj}{n} = \frac{ck(k+1)/2}{n^2}.$$

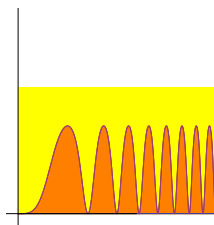
Taking the limit  $n \rightarrow \infty$  and using that  $k/n \rightarrow x$  shows that  $\int_0^x f(t) dt = cx^2/2$ .



**Linearity of the integral**  $\int_a^b f(t) + g(t) dt = \int_a^b f(t) dt + \int_a^b g(t) dt$  and  $\int_a^b \lambda f(t) dt = \lambda \int_a^b f(t) dt$ .

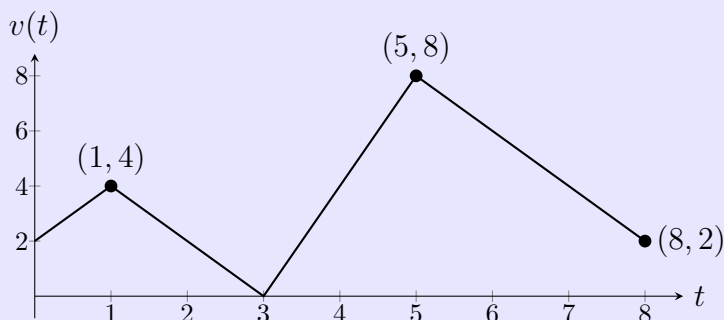
**Upper bound:** If  $0 \leq f(x) \leq M$  for all  $x$ , then  $\int_a^b f(t) dt \leq M(b-a)$ .

$\int_0^x \sin^2(\sin(\sin(t)))/x dt \leq x$ . **Solution.** The function  $f(t)$  inside the interval is non-negative and smaller or equal to 1. The graph of  $f$  is therefore contained in a rectangle of width  $x$  and height 1.



# Homework due 3/27/2024

**Problem 23.1:** Below is the graph of the velocity of a bee traveling from a clover to a hive. Find the exact distance traveled by the bee between  $t = 1$  and  $t = 8$ .

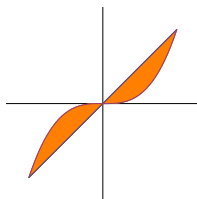


**Problem 23.2:** Let's look at the function  $f(x) = \sin(x)$  on  $[0, \pi]$ .

a) Approximate the integral  $\int_0^{\pi/2} \sin(x) dx$  using a Riemann sum with  $\Delta x = \pi/4$ .

b) Approximate the integral  $\int_0^{\pi/2} \sin(x) dx$  using the Riemann sum with  $\Delta x = \pi/6$ .

**Problem 23.3:** The region enclosed by the graph of  $x$  and the graph of  $x^5$  has a propeller type shape. Approximate its area by a Riemann sum using a Riemann sum with  $\Delta x = 1/4$ . It is your job to find  $a, b$  and  $n$  as well as the points  $x_k = a + k(b-a)/n$ .



**Problem 23.4:** Explain each rule with a picture:

- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$ .
- $\int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b (f(x) - g(x)) dx$ .
- $\int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx$ .

**Problem 23.5:** In this problem, it is crucial that you plot the function first. Split the integral up into parts. Find  $\int_{-1}^4 f(x) dx$  for  $f(x) = |x - |x - 2||$ . As in 23.1 do not use a Riemann sum here. You can compute the value exactly.