INTRODUCTION TO CALCULUS

MATH 1A

Unit 21: Finding Roots

21.1. Finding roots using the intermediate value theorem did not involve any differentiability. It worked for continuous functions. If a function is differentiable, we have more options. We can use linearization to find roots. We can also find intermediate points where the derivative is the average rate of change. The intermediate value will come handy when looking at the fundamental theorem.

21.2. Last time, we have seen to find roots of functions using a “divide and conquer” technique: start with an interval $[a, b]$ for which $f(a) < 0$ and $f(b) > 0$. If $f((a+b)/2)$ is positive, then use the interval $[a, (a+b)/2]$ otherwise $[(a+b)/2, b]$. After n steps, we are $(b-a)/2^n$ close to the root. If the function f is differentiable, we can do better and use the value of the derivative to get closer to a point $y = T(x)$. Lets find this point y. If we draw a tangent at $(x, f(x))$ and intersect it with the x-axes, then

$$f'(x) = \frac{f(x) - 0}{x - T(x)} .$$

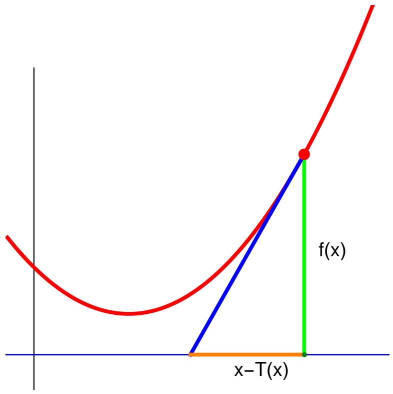
Now, $f'(x)$ is the slope of the tangent and the right hand side is ”rise” over ”run”. If we solve for $T(x)$, we get

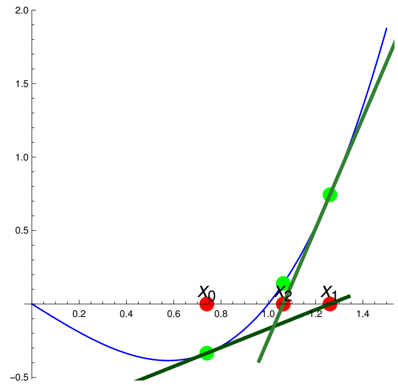
Definition: The **Newton map** of a function f is defined as

$$T(x) = x - \frac{f(x)}{f'(x)} .$$

21.3. The **Newton's method** applies this map a couple of times until we are sufficiently close to the root: start with a point x, then compute a new point $x_1 = T(x)$, then $x_2 = T(x_1)$ etc.

If p is a root of f such that $f'(p) \neq 0$, and x_0 is close enough to p, then $x_1 = T(x), x_2 = T^2(x)$ converges to the root p.





If $f(x) = ax + b$, we reach the root in one step.

If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$. We get exponentially fast to the root 0.

The Newton method converges extremely fast to a root $f(p) = 0$ if $f'(p) \neq 0$. In general, the number of correct digits double in each step.

In 4 steps we expect to have $2^4 = 16$ digits correct. Having a fast method to compute roots is useful. For example, in computer graphics, where things can not be fast enough. We will explore a bit in the lecture how fast the method is.

Lets compute $\sqrt{2}$ to 12 digits accuracy. We want to find a root $f(x) = x^2 - 2$. The Newton map is $T(x) = x - (x^2 - 2)/(2x)$. Lets start with $x = 1$.

$$\begin{aligned} T(1) &= 1 - (1-2)/2 = 3/2 \\ T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\ T(17/12) &= 577/408 \\ T(577/408) &= 665857/470832 \;. \end{aligned}$$

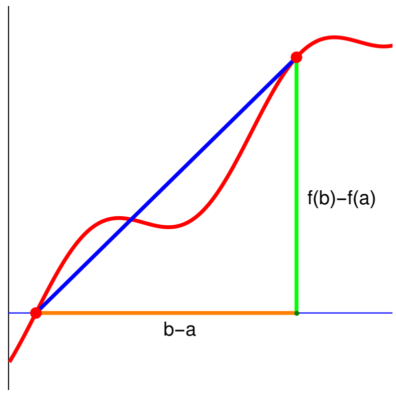
This is already $1.6 \cdot 10^{-12}$ close to the real root! 12 digits, by hand!

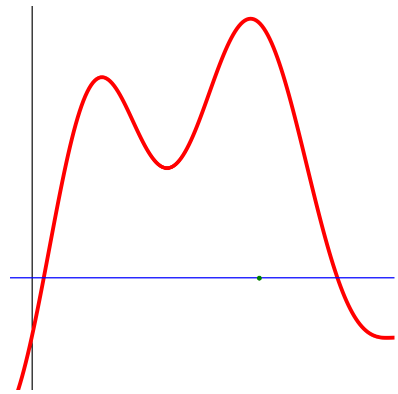
Mean value theorem

21.4. Unlike the intermediate value theorem which applied for continuous functions, the **mean value theorem** involves derivatives. Also here, we assume that f is differentiable unless specified. The mean value theorem is a consequence of the intermediate value theorem. It tells that the average rate of change is matched by an instantaneous rate of change somewhere.

Mean value theorem: Assume f is differentiable on $[a, b]$. Then there is a point x such that

$$f'(x) = \frac{f(b) - f(a)}{b - a} .$$





Here are a few examples which illustrate the theorem:

Verify with the mean value theorem that the function $f(x) = x^2 + 4\sin(\pi x) + 5$ has a point where the derivative is 1.
Solution. Since $f(0) = 5$ and $f(1) = 6$ we see that $(f(1) - f(0))/(1 - 0) = 5$.

A biker drives with velocity $f'(t)$ at position $f(b)$ at time b and at position a at time a. The value $f(b) - f(a)$ is the distance traveled. The fraction $[f(b) - f(a)]/(b - a)$ is the average speed. The theorem tells that there was a time when the bike had exactly the average speed.

21.5. Proof of the theorem: the function $h(x) = f(a) + cx$, where $c = (f(b) - f(a)/(b - a)$ also connects the beginning and end point. The function $g(x) = f(x) - h(x)$ has now the property that $g(a) = g(b)$. If we can show that for such a function, there exists x with $g'(x) = 0$, then we are done. By tilting the picture, we have reduced it to a statement seen before:

Rolle's theorem: If $f(a) = f(b)$ then f has a critical point in (a, b).

Proof: If it were not true, then either $f'(x) > 0$ everywhere implying $f(b) > f(a)$ or $f'(x) < 0$ implying $f(b) < f(a)$.

Show that the function $f(x) = \sin(x) + x(\pi - x)$ has a critical point $[0, \pi]$.
Solution: The function is differentiable and non-negative. It is zero at $0, \pi$. By Rolle's theorem, there is a critical point.

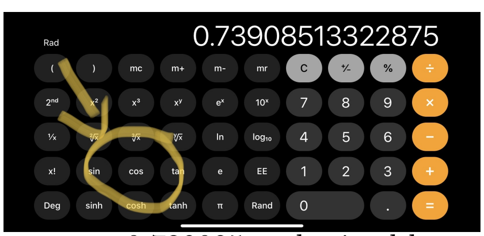
Verify that the function $f(x) = 2x^3 + 3x^2 + 6x + 1$ has only one real root. **Solution:** There is at least one real root by the intermediate value theorem: $f(-1) = -4, f(1) = 12$. Assume there would be two roots. Then by Rolle's theorem there would be a value x where $g(x) = f'(x) = 6x^2 + 6x + 6 = 0$. But there is no root of g. [The graph of g minimum at $g'(x) = 6 + 12x = 0$ which is 1/2 where $g(1/2) = 21/2 > 0$.]

Homework, due 3/22/2024

Problem 21.1: Get the Newton map $T(x) = x - f(x)/f'(x)$ for:
a) $f(x) = (x-2)^2$
b) $f(x) = e^{5x}$
c) $f(x) = 2e^{-x^2}$
d) $f(x) = \cot(x)$.

Solution:
a) $(x+2)/2$.
b) $x - 1/5$.
c) $x + 1/(2x)$.
d) $x + \cos(x)\sin(x)$.

Problem 21.2: The function $f(x) = \cos(x) - x$ has a root between 0 and 1. Starting with $x = 1$, perform the first Newton step.



Compare with the root $x = 0.739085...$ obtained by punching "cos" again and again

Solution:
$T(1) = 1 + (\cos(1) - 1)/(\sin(1) + 1) = 0.750365$.

Problem 21.3: We want to find the square root of 102. We have to solve $\sqrt{102} = x$ or $f(x) = x^2 - 102 = 0$. Perform a Newton step starting at $x = 10$.

Solution:
The step is $T(x) = x - (x^2 - 102)/(2x)$. This is $10 + 1/10 = 11/10$. The second step gives $20401/2020 = 10.0995$. The actual value is 10.0995.

Solution:
$f'(x) = \cos(x)/x - \sin(x)/x^2$ which satisfies $f'(\pi/2) = -4/\pi^2$. We have $f(\pi/2)/f'(\pi/2) = -\pi/2$. So that $T(\pi/2) = \pi$.

Problem 21.4: Find the Newton step $T(x) = x - f(x)/f'(x)$ in the case $f(x) = 1/x$. What happens if you apply the Newton steps starting with $x = 1$? Does the method converge?

Solution:
$T(x) = 2x$. We have $T^n(x) = 2^n x$. The point diverges to ∞.

Problem 21.5: We look at the function $f(x) = x^{10} + x^4 - 20x$ on the positive real line. Verify that the **mean value theorem** on $(1, 2)$ assures the there exists x, where $g(x) = f'(x) - [f(2) - f(1)] = f'(x) - 1018$. Now use a single Newton step starting with 1.5 to find an approximate solution to $g(x) = 0$.

Solution:
We have $f(2) - f(1) = 1018$. We need to solve $g(x) = f'(x) - 1018 = 0$. The Newton step is $1.5 - g(1.5)/g'(1.5) = 1.77428$.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024