INTRODUCTION TO CALCULUS

MATH 1A

Unit 17: Chain Rule

17.1. For the derivative of the composition of functions like $f(x) = \sin(x^7)$, we can not use the product rule. The functions don't hold hands like in a product, they are "chained" in the sense that we evaluate first x^7 then apply the sin function to it. In order to differentiate, take the derivative of the x^7 then multiply this with the derivative of the function sin evaluated at x^7. The answer is $\cos(x^7)7x^6$. Here is the rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \; .$$

17.2. The chain rule follows from the following identity that is true for **all** functions f, g such that $h, H = g(x+h) - g(x) > 0$. (If $g(x+h) = g(x)$, we just would get zero on the left.)

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x)+H) - f(g(x))]}{H} \cdot \frac{H}{h} \; .$$

If f, g are differentiable we can take the limit $h \to 0$, which gives $H \to 0$. The first part goes to $f'(g(x))$ and the second factor goes to $g'(x)$.

17.3. Let us look at some examples.

Problem: Find the derivative of $f(x) = (4x^2 - 1)^{17}$.
Solution The inner function is $g(x) = 4x^2 - 1$ with derivative $8x$. We get $f'(x) = 17(4x - 1)^6 \cdot 8x$.
Remark. Expansion of $(4x^2 - 1)^{17}$ would have avoided the chain rule. But we would have been using a **pain rule**.

Problem: Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution**: applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

For linear functions $f(x) = ax+b, g(x) = cx+d$, the chain rule can readily be checked: we have $f(g(x)) = a(cx+d)+b = acx+ad+b$ which has the derivative ac. This agrees with the definition of f times the derivative of g. Using this and linearization can prove the chain rule too.

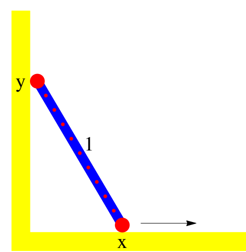
$f(x) = \sin(x^2+3)$. Then $f'(x) = \cos(x^2+3)2x$.

$f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x)$.

The name "chain rule" is because we can chain even more functions together:

Problem: Let us compute the derivative of $\sin(\sqrt{x^5-1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5 - 1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5-1}$ gives $\cos(\sqrt{x^5-1})\frac{d}{dx}\sqrt{x^5-1}$. Apply now the chain rule again for the derivative on the right hand side.

Example: remember the **falling ladder problem**, where a ladder of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0,y)$ on the wall with $(x,0)$ on the floor. We want to express y as a function of x. We have $y = f(x) = \sqrt{1-x^2}$. Taking the derivative, assuming $x' = 1$ we get $f'(x) = -2x/\sqrt{1-x^2}$. Infinite speed at the end. The ladder will definitely break the sound barrier.



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

Lets push that to the extreme and differentiate

$$f_{11}(x) = \exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(x))))))))))) .$$

Here is the poetic formula obtained when running this in Mathematica:

$$\mathrm{D[Last[NestList[Exp, x, 11]], x]}$$

$$\exp\left(e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{x}}}}}}} + e^{e^{e^{e^{e^{e^{x}}}}}} + e^{e^{e^{e^{e^{x}}}}} + e^{e^{e^{e^{x}}}} + e^{e^{e^{x}}} + e^{e^{x}} + e^{x} + x \right)$$

Note the exponential at the left. This can also be written as a product. In class we broke the record and wrote down the derivative of a chain of 12 exponentials and because $f_n' = f_n f_{n-1}'$ we have $f_{12}' = f_{12}f_{11}f_{10}f_9f_8f_7f_6f_5f_4f_3f_2f_1$.

Problem: Find the derivative of $1/\sin(x)$ using the quotient rule as well as using the chain rule using the function $g(x) = 1/x$. **Solution** $-\cos(x)\cdot 1/\sin^2(x)$.

Homework: Due Wed, 3/6/2024

Problem 17.1: Find the derivatives of the following functions:
a) $f(x) = xe^{-x^2}$
b) $f(x) = \ln(\ln(x))$
c) $f(x) = \cot(x^{17})$
d) $f(x) = x/(4 + x^2)$
e) $(\sin(x) + \cos(x))^{-3}$

Problem 17.2: Find the derivatives of the following functions at $x = 1$.
a) $\sqrt{x^2 - 1}$
b) x^x.
c) $(1 + x^3)^{100}$
d) $\sin(\sin(\sin(x)))$
e) $\sin(\sin(\sin(\sin(\sin(x)))))$.

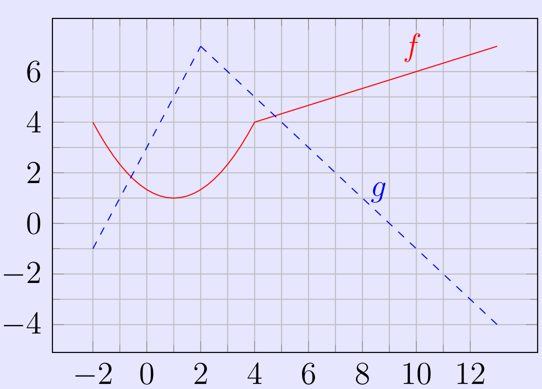
Problem 17.3: We also need to practice taking derivatives with respect to variables that are different from x.
a) Find the derivative of $e^{-t^2} \sin(t^4)$ with respect to t.
b) Find the derivative of $\cos(\sin(\cos(u^2)))$ with respect to u.

Problem 17.4: Lets go back to some rule we have already seen earlier.
a) Verify that

$$\frac{d}{dx} f(cx + b) = cf'(cx + b)$$

b) $\frac{d}{dx}[\ln(6x + 3) + \sin(3x + 5) + \cos(11x + 3)]$.

Problem 17.5: The following graph shows functions f and g. If $F(x) = f(g(x))$, find $F'(1)$ and $F'(8)$.



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