INTRODUCTION TO CALCULUS

MATH 1A

Unit 13: Global maxima

13.1. In applications, the domain of the function can be limited. For example, if we want to find the rectangle of width x and length y that maximizes the area xy given a circumference of $2x+2y=4$, we have to find the maximum of $f(x)=x(2-x)=2x-x^2$. But obviously, the width can not be negative, nor can it be larger than 2 without $y=2-x$ becoming negative. We need to maximize on the closed interval $[0,2]$.

13.2. A point $x \in [a,b]$ is a **global maximum** if there is no point y in $[a,b]$ for which $f(y) > f(x)$. The point x is a **global minimum** if x is a global maximum for $-f$. Here is a theorem of Bolzano: [1]

Extreme value theorem: A continuous function f on $[a,b]$ has a global maximum and a global minimum.

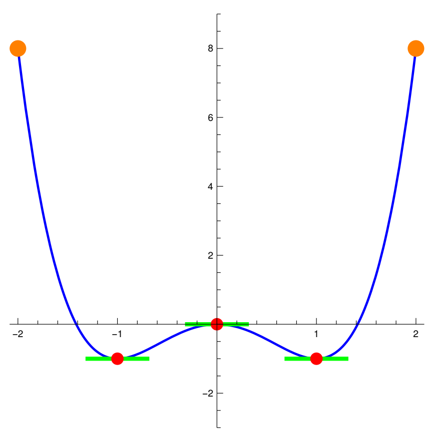

FIGURE 1. A prototype example $f(x)=x^4-2x^2$. There are two local minima $-1,1$ and 3 local maxima. The minima are also global. Only the boundary points are global maxima. Note that $f'(x)$ does not need to be zero at the boundary points. This example is iconic and a Goldstone boson picture used to explain **spontaneous symmetry breaking** in physics.

[1] We do not use the term "absolute maximum", as it suggests to look a maximum of $|f|$.

13.3. Problem: Find the global maximum and minimum of $f(x) = x^4 - 2x^2$ on $[-2, 2]$. **Solution**: $f'(x) = 4x^3 - 4x = 0$ for $x = -1, 0, 1$. The second derivative $f''(x) = 12x^2 - 4$ is positive at $x = -1, 1$ and negative at $x = 0$. We have two local minima $-1, 1$ and one maximum 0. Include the boundary points $-2, 2$. The total list of points to consider is $\{-2, -1, 0, 1, 2\}$. The function values are $\{8, -1, 0, -1, 8\}$. The points $\{-1, 1\}$ are global minima, the points $-2, 2$ are global maxima.

To find a global maximum of f on $[a, b]$, make a list of local maxima in the interior (a, b) using the first or second derivative test, then collect the boundary points as candidates. Among this combined list, chose where f is maximal.

13.4. Back to the rectangle problem with $f(x) = x(2-x) = 2x - x^2$ on $[0, 2]$. We find the local maxima using the second derivative test $f'(x) = 2 - 2x = 0$ for $x = 1$. The boundary point values are $f(0) = 0$ and $f(2) = 0$. The later are the global minima.

13.5. Here is a similar but a bit more complex problem:

Which isosceles triangle of height h and base $2x$ and area $xh = 1$ has minimal circumference $2x + 2\sqrt{x^2 + h^2}$?

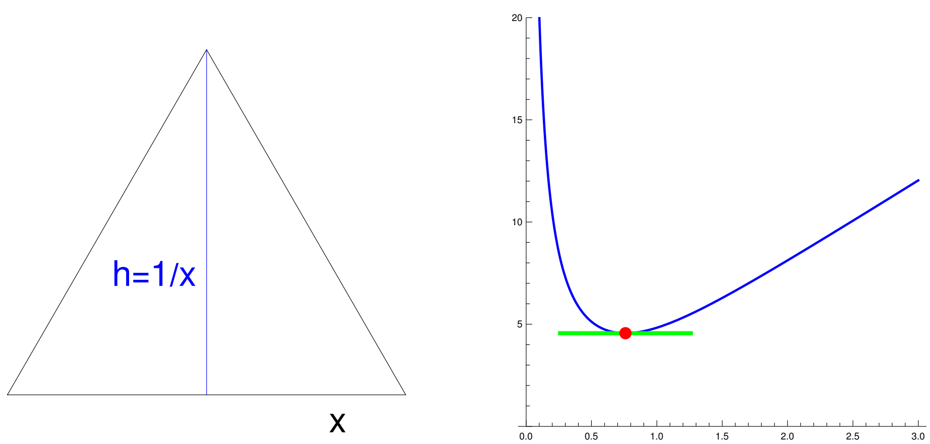

FIGURE 2. A function $f(x) = 2x + 2\sqrt{x^2 + 1/x^2}$ gives the circumference of a triangle with base $2x$ and height $h = 1/x$. We want to find the minimal circumference. f is defined on $(0, \infty)$. There is a unique minimum. There are no boundary points to consider. This example is a special case of the **isoperimetric inequality**: a **n-gon** of area 1 with minimal circumference must be a regular n-gon. Symmetry rules!

13.6. We have to extremize the function $f(x) = 2x + 2\sqrt{x^2 + 1/x^2}$. The base length can not be negative, nor zero so we have to look at the problem on $(0, \infty)$. There are no boundary points to consider so that only candidates for minima are places, where $f'(x) = 0$. The only positive solution of of $f'(x) = 0$ is $x = 1/3^{1/4}$. This means $h = 3^{1/4}$. One can check that $x^2 + h^2 = 4x^2$ so that this is an equilateral triangle.

13.7. About the proof of the extreme value theorem: pick a point x_1. If there is no point with a larger value, it is the global maximum. If not, there is a point x_2, where $f(x_2) > f(x_1)$. If there is no other point with larger value, then x_2 is the global maximum. The alternative is that there is an other point x_3 with $f(x_3) > f(x_2)$. Continuing like this we either end up at a global maximum or then find a sequence of numbers x_n with increasing f values. Now split $I_1 = [a, b]$ into two intervals of equal length $(b - a)/2$. The sequence x_n has to visit one (or both) of them infinitely often. Pick such an interval I_2 and renumber the x_k which hit that interval. Again split the interval into two intervals of length $(b - a)/4$ and pick one I_3 for which there are infinitely many points x_k. We have now a nested sequence of intervals of smaller and smaller length. The intersection of all these intervals is a single point $[x, x]$. This point is a global maximum. The same argument shows that if x_n would be unbounded, then the function would not be continuous at x.

Homework: Due Friday 2/23/2024

Problem 13.1: Find the global maxima and minima of $f(x) = x^3 - 6x^2 + 9x + 7$ on $[-2, 6]$.

Solution:
the critical points are $1, 3$. The second derivative is negative at -1 and positive at 3. The point 1 is a local max, the point 3 is a local min. We also need to look at the boundary points. We have $f(-2) = -43$ and $f(6) = 61$. Compare this with the local max $f(1) = 11$ and the local min $f(3) = 7$ to get that -2 is the global min and 6 is the global max.

Problem 13.2: Find all the global maxima and minima of $f(x) = 3x^{2/3} - x$ on $[-1, 1]$.

Solution:
There is only one critical point at $x = 8$ which is outside the interval. The boundary points have values $f(-1) = 4$ and $f(1) = 2$. There is also a point $x = 0$, where the derivative $f'(x) = 2x^{-1/3} - 1$ is not defined. This point has to be considered separately. Indeed $f(0) = 0$ is a global minimum. The point -1 is the global maximum.

Problem 13.3: Find all the global maxima and minima of $f(x) = t^{-1} + 2t^{-2}$ on $[1, \infty)$.

Solution:
The derivative is $f'(t) = -t^{-2} - 4t^{-3}$. It is zero at $t = -4$ which is outside the interval under consideration. The point $x = 1$ is a boundary point where $f(1) = 3$. This is the global maximum. There is no global minimum, not even a local minimum.

Problem 13.4: Find all the global maxima and minima of $|\ln(x + 3)|$ on $[-3, \infty]$.

Solution:
The function is zero at $x = -2$ and positive else. This is the point where the absolute value kicks in and where we have a global minimum. The function needs to be considered on the interval $x > -2$ first where $\ln(x + 3)$ is positive. There, the derivative is $1/(x + 3)$ which is positive and never zero. On the interval $-3, -2$ the derivative is $-1/(x + 3)$ which is negative and never zero. There is no global max nor a global min.

Problem 13.5: Does there exist an example of a continuous function on $[-2, 2]$ with the given properties or not? If yes, pick an example from the given list A-D.
A) $f(x) = x^3$
B) $f(x) = 3$
C) $f(x) = x^2$
D) $f(x) = -x^2$

Question	Yes	No	Example if yes A-D
a) f has a global max but no global min			
b) f has only global max and global min			
c) f has a two global max and one global min			
d) f has a one global max and one global min			

Solution:
a) No, because of the extreme value theorem.
b) yes, Take $x = 5$ for example. Also x^2 has only global maxima and global minima as as candidates for maxima and minima. If the interpretation is that every point is both a global max and a global min, then $x = 5$ is the right answer. c) yes, Take x^2
d) yes, take $x^3 4$.

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