INTRODUCTION TO CALCULUS

MATH 1A

Unit 10: Infinity

10.1. This lecture is about infinity. The main point is that Hospital's rule for the **indefinite form** "0/0" works also for the **indefinite form** "∞/∞" as well as when $p = \infty$. Given a function $f(x)$, we can look how $f(x)$ grows when $x \to \infty$. If there is a limit for $x \to \infty$, we have a **horizontal asymptote**. For example $\lim_{x\to\infty} \arctan(x) = \pi/2$. We can also reach infinity vertically. If $\lim_{x\to p} f(x)$ does not exist, there might be a **vertical asymptote**. The function $f(x) = \frac{x^2+1}{x^2-1}$ for example has a horizontal asymptote $y = 1$ as l'Hospital gives $\lim_{x\to\infty} f(x) = 1$. and vertical asymptotes at $x = 1$ and $x = -1$. If $\lim_{x\to p} f(x) = \infty$ and $\lim_{x\to p} g(x) = \infty$ we can ask what happens with the limit of $\lim_{x\to p} f(x)/g(x)$. Again, this can be done with Hospital.

Hospital's rule. If f, g are differentiable and $\lim_{x\to p} f(p) = \lim_{x\to p} g(p) = \infty$ and $\lim_{x\to p} g'(x) \neq 0$, then

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \lim_{x\to p} \frac{f'(x)}{g'(x)} .$$

Find $\lim_{x\to\infty}(7x^2 + x + 1)/(3x^2 - 1)$. **Solution.** We check to have an indefinite form ∞/∞. Differentiate both nominator and denominator to get $\lim_{x\to\infty}(14x + 1)/6x$. Having again an indefinite form ∞/∞, we send it again to the Hospital. The answer is $14/6 = 7/3$.

Find $\lim_{x\to 0} \log|7x|/\log|3x|$. This is an indefinite form ∞/∞. We can use l'Hospital and see $\lim_{x\to 0}(7/7x)/(3/3x)$ which can be simplified to 1 for $x \to 0$.

10.2. About the proof: the case when both sides converge to infinity can be reduced to the 0/0 case by writing $A = f/g = (1/g(x))/(1/f(x))$. Use l'Hospital and take the derivative on both sides simplifies to $(g'/f')A^2$. Solving for A gives $A = f'(p)/g'(p)$.

Problem: Lets look at the limit $\lim_{x\to\pi/2} \tan(3x)/\tan(7x)$. First check this is an indefinite form ∞/∞. Now take the derivatives on both sides: $\lim_{x\to\pi/2}(3/\cos^2(3x))/(7/\cos^2(7x)) = 3/7$.

Homework

Problem 10.1: Lets look at the functions $f(x) = \ln(x)$ and $g(x) = x$. In order to see which function grows faster, we study the limit $\lim_{x\to\infty} f(x)/g(x)$.
a) What is $f(x)/g(x)$ for $x = e^{10}$ and $x = e^{1000}$?
b) Compute the limit $\lim_{x\to\infty} f(x)/g(x)$ for $x \to \infty$ using l'Hospital.

Solution:
a) $f(e^{10}) = 10$ and $f(e^{100}) = 100$. So $f/g = 10/e^{10}$ in the first case and $f/g = 100/e^{100}$ in the second case.
b) The limit is 0. l'Hospital gives $\lim_{x\to\infty} = (1/x)/1 \to 0$.

Problem 10.2: Let us introduce the notation $f(x) \ll g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 0$. The meaning is that $f(x)$ **grows asymptotically slower** than $g(x)$. For example, $\sqrt{x}\sin(x) \ll x$ because $\sqrt{x}\sin(x)/x \to 0$ for $x \to \infty$.
a) Rank the functions $e^x, x^x, \ln(x), \sqrt{x}, x, x^2$ with that order notation. Which one grows slowest, which is next etc, until which is grows fastest?
b) Produce one single graphics that shows the graphs of all these 6 functions, plotted on the interval $[0, 3]$.

Solution:
$\ln(x) \ll \sqrt{x} \ll x \ll x^2 \ll e^x \ll x^x$.

Problem 10.3: Find the following limits involving the indefinite form ∞/∞:
a) $\lim_{x\to 0} \cot(x)/\cot(3x)$.
b) $\lim_{x\to\infty} \frac{3x^2+1}{4x^2+100}$.
c) $\lim_{x\to 0} \sqrt{\log(3x)}/\sqrt{\log(2x)}$.
d) Find $\lim_{x\to\infty}(x^2 + x - 1)/\sqrt{5x^4 + 1}$.
Hint to to c) and d): First square the expression and find the limit, then take the root of the result.

Solution:
a) l'Hospital gives the limit $1/\sin^2(x)/(3/\sin^2(3x) = \sin^2(3x)/(3\sin^2(x))$. First break it apart to get $\lim_{x\to 0}\sin(3x)/\sin(x)\lim_{x\to 0}\sin(3x)/\sin(x)/3 = 3*3/3 = 3$.
b) Bring it twice to the hospital to get 6/8.
c) Compute the limit of the square. The square is $\log(3x)/\log(2x)$ which by l'Hospital has the limit 1. So also the square root is 1. d) The squared version has the limit $(4/20) = 1/5$. So, the limit is $1/\sqrt{5}$.

Problem 10.4: Use l'Hospital to compute the following routine limits $x \to \infty$ (we use for a change $\log = \ln$, which is the common notation in all computer programming languages and all higher mathematics).
a) $\log|x|/x$
b) $\log|5x|/\log|x|$.
c) $x^2/(1+x^2)$.
d) $\log|1+x|/\log|2+x|$.
e) $(e^x - 1)/(e^{2x} - 1)$

Solution:
In each case, differentiate. Note that $\frac{d}{dx}\ln(cx) = 1/x$.
a) 0
b) 1
c) 1
d) 1 e) 0 This can also be done by factoring out and see it as $1/(e^x + 1)$.

Problem 10.5: We have $nx = x + x + x + \cdots + x$ and $x^n = x * x * x * \cdots * x$. In computer science, the Knuth arrow notation write this as $x \uparrow n$. The number 3^4 for example is $3 * 3 * 3 * 3 = 256$. Knuth then writes $x \uparrow\uparrow n = x \uparrow x \uparrow \ldots \uparrow x$ meaning to exponentiate n times. For example, $x \uparrow\uparrow 4 = x^{x^{x^x}}$.
a) We have already studied $f(x) = x^x = x \uparrow\uparrow 2$. Lets look at $g(x) = x \uparrow\uparrow 3 = x^{(x^x)}$. Compute the numbers $g(1), g(2), g(3)$ and if you have the tools, compute $g(4)$.
b) What is the limit $\lim_{x\to\infty} f(x)/g(x) = x^x/x^{(x^x)}$? You can use l'Hospital, but you need an idea.

Solution:
a) $1^{1^1} = 1$,
$2^{2^2} = 2^4 = 16$,
$3^{3^3} = 3^27 = 7625597484987$,
$4^{4^4} = 134078079299425970995740249982058461274793658205923933777235614437217640300735\ 46976801$
b) Write $x^x = y$, then compute the limit $y/x^y \leq y/2^y$ for $x > 2$. Since $y/2^y \to 0$, we know that the limit is zero.

Here are the first $x \uparrow\uparrow n$ for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. We don't write the brackets. The meaning is always that you put the brackets from the right, like $x^{x^x} = x^{(x^x)}$:

$$\left\{ x, x^x, x^{x^x}, x^{x^{x^x}}, x^{x^{x^{x^x}}}, x^{x^{x^{x^{x^x}}}}, x^{x^{x^{x^{x^{x^x}}}}}, x^{x^{x^{x^{x^{x^{x^x}}}}}}, x^{x^{x^{x^{x^{x^{x^{x^x}}}}}}}, x^{x^{x^{x^{x^{x^{x^{x^{x^x}}}}}}}}, \right\}$$

And here is $x \uparrow\uparrow 100$.

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And now lets get dizzy: Knuth defines $x \uparrow\uparrow\uparrow n$ as $x \uparrow\uparrow x \uparrow\uparrow x \dots \uparrow\uparrow x$. For example, $x \uparrow\uparrow\uparrow 2 = x \uparrow\uparrow x$.

Now

$$10 \uparrow\uparrow\uparrow 2 = 10 \uparrow\uparrow 10 = 10^{10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}}}$$

There is no way that we can even write down this number. We estimate to have 10^{80} elementary particles in the universe.

Now wrap your head around $10 \uparrow\uparrow\uparrow\uparrow 2 = 10 \uparrow\uparrow\uparrow 10 = 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10$.
We do not have even to go to infinity to become insane. Even large finite numbers can drive you nuts. Big time.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024