INTRODUCTION TO CALCULUS

MATH 1A

Unit 9: Hospital

9.1. Hospital's rule is a fantastic tool. It allows to compute limits. [1] It is a miracle procedure and the answer to all our prayers to save us from dreadful limit computations!

Hospital's rule. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)} \, .$$

Lets see how it works in examples:

The fundamental theorem of trigonometry:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 \, .$$

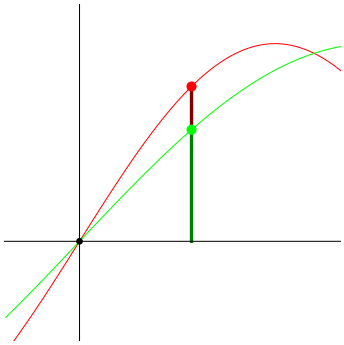
Note that this does not replace the derivation because it is equivalent to $\sin' = \cos$!

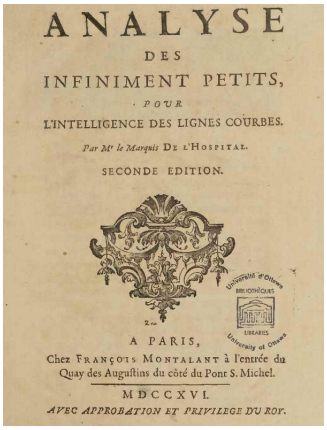
9.2. The proof of the rule is very simple: since $f(p) = g(p) = 0$ we have for the average rate of changes $Df(p) = (f(p+h) - f(p))/h = f(p+h)/h$ and $Dg(p) = (g(p+h) - g(p))/h = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$. So, the **quantum l'Hospital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} \, .$$

Now take the limit $h \to 0$. On the left we get $\lim_{h \to p} f(x)/g(x)$ by definition. On the right we get $f'(p)/g'(p)$ by definition. Voilà!

[1] Hospital is is easier to write and remember than Hôpital. Bring f to the hospital!





Problem. Find the limit $f(x) = (\exp(2x) - 1)/x$ for $x \to 0$.
Answer. The rule gives 2.

Problem. Find the limit $f(x) = \sin(100x)/\sin(101x)$ for $x \to 0$.
Answer. The rule 100/101.

9.3. The "first calculus book" was "Analyse des Infiniment Petits pour l"intelligence des Lignes Courbes" appeared in 1696. It was written by **Guillaume de l'Hospital** and has about 50-100 pages. [2] The mathematical content is mostly due to **Johannes Bernoulli**. The book remained the standard for a century.

9.4. Sometimes, we have to administer l'Hospital twice:

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x\to p} \frac{f(x)}{g(x)} = \lim_{x\to p} \frac{f''(x)}{g''(x)}$ if $g''(p) \neq 0$.

Problem: What do you get if you apply l'Hospital to the limit $[f(x + h) - f(x)]/h$ as $h \to 0$?
Answer: Differentiate both sides with respect to h! And then feel awesome!

What is the limit $\lim_{x\to 0} |x|^x$? This will provide the best answer to the question **What is 0^0?**

Find the limit $\lim_{x\to 2} \frac{x^2-4x+4}{\sin^2(x-2)}$.
Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(0) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.

[2] Stewart's book with 1200 pages probably contains about 4 million characters, about 12 times more than l'Hospital's book. The OCR text of l'Hospital's book of 200 pages has 300'000 characters.

Homework

Problem 9.1: For the following functions, find the limits as $x \to 0$ using Hospital:
a) $\sin(7x)/(5x)$
b) $(\exp(16x)-1)/(\exp(17x)-1)$
c) $\sin^2(8x)/\sin^2(5x)$
d) $\frac{\tan(4x)}{3x}$
e) $\sin(\sin(11x))/x$.

Solution:
All with l'Hospital:
a) 7/5
b) 16/17
c) $(8/5)^2$
d) 4/3.
e) 11.

Problem 9.2: **Luna**, a new math chatbot, teaches itself limits but still makes mistakes and struggles with concepts. Please evaluate its answers to the right:

Problem	Luna's Reasoning
a) $\lim_{x\to\pi} \frac{\cos x}{x-\pi}$	$\lim_{x\to\pi} \frac{\cos x}{x-\pi}$ is by Hospital equal to $\lim_{x\to\pi} \frac{-\sin x}{1} = 0$
b) $\lim_{x\to 5} \frac{x-5}{\sqrt{x-5}} = 0$	Hospital: $\lim_{x\to 5} 1/(1/(2\sqrt{x-5})) = \lim_{x\to 5} 2\sqrt{x-5} = 0$
c) $\lim_{x\to 0} \frac{e^x-1}{x}$	$\lim_{x\to 0}(e^x-1) = 0$ and 0 over anything is 0, the limit is 0.
d) $\lim_{x\to 0} \ln\|x\| x = 0$	$\ln\|x\| \to -\infty$ and $x \to 0$. As it is not f/g, the limit DNE.
e) $\lim_{x\to 0} \ln\|2x\|/\ln\|x\|$	Hospital gives $\lim_{x\to 0}(1/2x)/(1/x) = \lim_{x\to 0} 1/2 = 1/2$.

Solution:
a) Hospital does not apply because it is not a $\frac{0}{0}$ limit.
b) 0/0 is an indefinite form and Hospital works. The argumentation is correct even so one could also just simplify.
c) This would have to be brought to the Hospital. The argumentation without it does not work and is also wrong.
d) If an expression does not match l'Hospital, it does not mean the limit does not exist. Indeed, we can write $(\ln|x|/(1/x)$ and use l'Hospital: $f'/g' = 1/x/(-1/x^2) = x$ converges to 0. e) This is just a calculation error. The result is 1 not 1/2.

Problem 9.3: Use l'Hospital to compute the following limits $x \to 0$:
a) $\lim_{x\to 0} x/\ln|x|$
b) $\ln|5x|/\ln|x|$.
c) $4\text{sinc}'(x) = 4(\cos(x)x - \sin(x))/x^2$
d) $\ln|1+x|/\ln|2+x|$.
e) $(e^x - 1)/(e^{2x} - 1)$

Solution:
a) 0. This is of the form $0/\infty$. There is no need for l'Hospital.
b) 1. The derivatives on the top and bottom are both $1/x$.
c) Take l'Hospital and cancel an x to get 0. d) No need for lHospital as the bottom is not 0. The result is 0.
e) A nice case for lHospital. The result is $1/2$.

Problem 9.4: We have seen how to compute limits with healing. Fix the broken bones by bringing them to the Hospital at $x \to 1$:
a) $\frac{x^{100}-1}{x^{22}-1}$.
b) $\frac{\tan^2(x-1)}{(\cos(x-1)-1)}$

Solution:
a) $100/22 = 50/11$.
b) Wse Hospital, then cancel out a $\sin(x-1)$ to get -2.

Problem 9.5: These problems need to be done during commercial breaks of the Super Bowl! If you fail to do so, you will be sent to the hospital by fierce 1a minions.
a) Find the limit $\lim_{x\to 0} \frac{x}{\tan(6x)}$.
b) Find the limit $\lim_{x\to 5} \frac{x^2-25}{x-5}$
c) Find the limit $\lim_{x\to 0} \frac{1-e^x}{x-x^3}$.
d) Find the limit $\lim_{x\to 0} \frac{\ln(1+9x)}{4x}$.
e) Find the limit $\lim_{x\to 1}(x^7 - 1)/(x^3 - 1)$.

Solution:
a) With Hospital 1/6.
b) Can also be done by healing. 10.
c) Nice case for Hospital: -1.
d) Nice case for Hospital: 9/4.
e) Also perfect case for Hospital 7/3.

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