

# INTRODUCTION TO CALCULUS

MATH 1A

## Unit 7: Basic Derivatives

7.1. We have already seen the key power identity:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

7.2. As a general rule we can state already now is that for any constant  $c$

$$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$$

In short  $(cf)' = cf'$ . The reason is that for every  $h$ , we have the property that the average rate of change  $(f(x+h) - f(x))/h$  has the property that **it plays nice with linearity**. This property goes over to the limit when the Planck constant  $h$  goes to zero.

7.3. An other general rule is the **addition rule**:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

7.4. Now we can compute the derivative of **any polynomial**: for example: lets compute the derivative of  $4x^3 + 3x^2 + x + 3$ . The answer is  $12x^2 + 6x + 1$ .

7.5. We have defined  $f(x) = e^x$  as the compound interest limit  $(1+h)^{x/h}$  for  $h \rightarrow 0$ .<sup>1</sup> Since we can check that  $(f(x+h) - f(x))/h = f(x)$  for any  $h > 0$ , the exponential function also has the property  $\frac{d}{dx}e^x = e^x$  and more generally

$$\frac{d}{dx}e^{cx} = ce^{cx}.$$

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<sup>1</sup>This limit  $e^x$  exists: use the squeeze theorem for  $g(x) \leq f(x) \leq h(x)$  with the decreasing  $g(h) = (1+h)^{x/h}$  and increasing  $k(h) = (1+h)^{x/h}(1+h)$  using that for  $h \leq 1$  one has  $k(h) - g(h) = h(1+h)^{x/h} \leq h2^x$  which converges to 0 for  $h \rightarrow 0$ .

**7.6.** If you increase  $x$  by a factor  $c$  faster, also the slope gets scaled by  $c$ . In general

$$f'(cx) = cf'(x)$$

You generalize this slightly in the homework. Later we will learn it as a special case of the chain rule.

**7.7.** In order to see the derivatives of the trig functions, remember first the **fundamental theorem of trigonometry** and companion identity:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1, \lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = 0.$$

Both these limits follow from the squeeze theorem. (See the short video). If we divide one of the addition formulas for trig functions by  $h$

$$\sin(x + h) - \sin(x) = \cos(x) \sin(h) + \sin(x)(\cos(h) - 1)$$

we get  $\cos(x)$  in the limit  $h \rightarrow 0$ . If we divide the second addition formula

$$\cos(x + h) - \cos(x) = \cos(x)(\cos(h) - 1) - \sin(x) \sin(h)$$

by  $h$  and take the limit we get  $-\sin(x)$ . We have shown

$$\frac{d}{dx} \sin(x) = \cos(x), \frac{d}{dx} \cos(x) = -\sin(x)$$

**7.8.** In the homework for today you have shown from the definition:

$$\lim_{h \rightarrow 0} \left[ \frac{1}{x+h} - \frac{1}{x} \right] / h = -\frac{1}{x^2}.$$

You have also seen in the homework that

$$\lim_{h \rightarrow 0} [\sqrt{x+h} - \sqrt{x}] / h = \frac{1}{2\sqrt{x}}.$$

These are special cases for the following formula for  $x^n$ . Indeed, for **any real number**  $n$  also negative ones, we have

$$\frac{d}{dx} x^n = nx^{n-1}.$$

**7.9.** By the way, we can see this also from writing  $x^n = e^{n \ln(x)}$ , using the chain rule (covered later in the course) and using  $\frac{d}{dx} \ln(x) = 1/x$  which we are going to look at next.

**7.10.** In the last worksheet, we simplified  $\lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h}$  to  $\lim_{h \rightarrow 0} \frac{\ln(1+h/x)}{h}$ . We need to get the limit  $\ln(1+h)/h$ . Since the limit  $h \rightarrow 0$  of  $(1+h)^{1/h}$  gives  $e$  by definition,  $\ln(1+h)/h \rightarrow 1$  and so  $\ln(1+h/x)/h \rightarrow 1/x$ .

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

## HOMEWORK

This homework is due on Wednesday 2/7, 2024.

**Problem 7.1:** Compute the following derivatives.

- $2x^5 + 3x^2 + 4x + 8$ .
- $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$ .
- $1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120$
- $\sqrt{x} + x^{3/2} + x^{5/2}$
- $(1 + x)(1 + x + x^2 + x^3 + x^4)$

**Problem 7.2:** We have seen that  $\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$ .

- Why is  $\frac{d}{dx} \frac{1}{x-5} = -\frac{1}{(x-5)^2}$ ?
- Formulate a general rule  $\frac{d}{dx} f(x-a) = f'(\dots\dots\dots)$  which holds for any differentiable function  $f$ .
- Formulate a general rule  $\frac{d}{dx} f(cx+b) = f'(\dots\dots\dots)$  which holds for any differentiable function  $f$ .

**Problem 7.3:** Compute the following derivatives

- $4 \sin(3x) + 7 \cos(9x)$
- $\ln(x) + \ln(2x) + \ln(3x)$
- $10e^{11x} + 8e^{20x} - 20e^{100x}$
- $9x^2 + \frac{1}{x^7} + 2 \cos(3x) + \ln(7x) - e^x$ .
- $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ .

- Problem 7.4:**
- Compute the derivative of  $f(x) = \pi^x$ . You first might have to rewrite the function in a form which allows you to use rules you know.
  - What is the derivative of  $a^x$  in general if  $a > 0$  is an arbitrary number?

- Problem 7.5:**
- Compute the derivative of  $f(x) = \sqrt{3x+5}$  from the rules you know.
  - In order to appreciate what we have achieved, compute the limit

$$\lim_{h \rightarrow 0} [f(x+h) - f(x)]/h .$$

for the function  $f(x) = \sqrt{3x+5}$  the old way as in PSet 6.