INTRODUCTION TO CALCULUS

MATH 1A

Unit 7: Basic Derivatives

7.1. We have already seen the key power identity:

$$\frac{d}{dx}x^n = nx^{n-1} .$$

7.2. As a general rule we can state already now is that for any constant c

$$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$$

In short $(cf)' = cf'$. The reason is that for every h, we have the property that the average rate of change $(f(x+h) - f(x))/h$ has the property that **it plays nice with linearity**. This property goes over to the limit when the Planck constant h goes to zero.

7.3. An other general rule is the **addition rule**:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

7.4. Now we can compute the derivative of **any polynomial**: for example: lets compute the derivative of $4x^3 + 3x^2 + x + 3$. The answer is $12x^2 + 6x + 1$.

7.5. We have defined $f(x) = e^x$ as the compound interest limit $(1+h)^{x/h}$ for $h \to 0$. [1] Since we can check that $(f(x+h) - f(x))/h = f(x)$ for any $h > 0$, the exponential function also has the property $\frac{d}{dx}e^x = e^x$ and more generally

$$\frac{d}{dx}e^{cx} = ce^{cx}.$$

[1] This limit e^x exists: use the squeeze theorem for $g(x) \leq f(x) \leq h(x)$ with the decreasing $g(h) = (1+h)^{x/h}$ and increasing $k(h) = (1+h)^{x/h}(1+h)$ using that for $h \leq 1$ one has $k(h) - g(h) = h(1+h)^{x/h} \leq h2^x$ which converges to 0 for $h \to 0$.

7.6. If you increase x by a factor c faster, also the slope gets scaled by c. In general

$$f'(cx) = cf(cx)$$

You generalize this slightly in the homework. Later we will learn it as a special case of the chain rule.

7.7. In order to see the derivatives of the trig functions, remember first the **fundamental theorem of trigonometry** and companion identity:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \ , \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0 \ .$$

Both these limits follow from the squeeze theorem. (See the short video). If we divide one of the addition formulas for trig functions by h

$$\sin(x+h) - \sin(x) = \cos(x)\sin(h) + \sin(x)(\cos(h)-1)$$

we get $\cos(x)$ in the limit $h \to 0$. If we divide the second addition formula

$$\cos(x+h) - \cos(x) = \cos(x)(\cos(h)-1) - \sin(x)\sin(h)$$

by h and take the limit we get $-\sin(x)$. We have shown

$$\frac{d}{dx}\sin(x) = \cos(x), \ \frac{d}{dx}\cos(x) = -\sin(x)$$

7.8. In the homework for today you have shown from the definition:

$$\lim_{h\to 0} [\frac{1}{x+h} - \frac{1}{x}]/h = -\frac{1}{x^2} \ .$$

You have also seen in the homework that

$$\lim_{h\to 0} [\sqrt{x+h} - \sqrt{x}]/h = \frac{1}{2\sqrt{x}} \ .$$

These are special cases for the following formula for x^n. Indeed, for **any real number** n also negative ones, we have

$$\frac{d}{dx} x^n = n x^{n-1}.$$

7.9. By the way, we can see this also from writing $x^n = e^{n\ln(x)}$, using the chain rule (covered later in the course) and using $\frac{d}{dx}\ln(x) = 1/x$ which we are going to look at next.

7.10. In the last worksheet, we simplified $\lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h}$ to $\lim_{h\to 0} \frac{\ln(1+h/x)}{h}$. We need to get the limit $\ln(1+h)/h$. Since the limit $h \to 0$ of $(1+h)^{1/h}$ gives e by definition, $\ln(1+h)/h \to 1$ and so $\ln(1+h/x)/h \to 1/x$.

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Homework

This homework is due on Wednesay 2/7, 2024.

Problem 7.1: Compute the following derivatives.
a) $2x^5 + 3x^2 + 4x + 8$.
b) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$.
c) $1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120$
d) $\sqrt{x} + x^{3/2} + x^{5/2}$
e) $(1 + x)(1 + x + x^2 + x^3 + x^4)$

Solution:
a) $10x^4 + 6x + 4$.
b) $-\frac{1}{x^2} - \frac{2}{x^3} - \frac{3}{x^3}$
c) $1 + x + x^2/2 + x^3/6 + x^4/24$.
d) $\frac{1}{2\sqrt{x}} + \frac{3}{2}\sqrt{x} + \frac{5}{2}x^{3/2}$.
e) FOIL first out to get $1 + 2x + 2x^2 + 2x^3 + 2x^4 + x^5$. Now take the derivative $2 + 4x + 6x^2 + 8x^3 + 5x^4$.

Problem 7.2: We have seen that $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$.
a) Why is $\frac{d}{dx}\frac{1}{x-5} = -\frac{1}{(x-5)^2}$?
b) Formulate a general rule $\frac{d}{dx}f(x-a) = f'(..........)$ which holds for any differentiable function f.
c) Formulate a general rule $\frac{d}{dx}f(cx+b) = f'(.........)$ which holds for any differentiable function f.

Solution:
a) $f(x-a)$ is just a translate of the function. If we want to get the slope at $x-5$, move over 5, compute the derivative at x then translate back. Alternatively, we can call $x-a$ a new variable t. Taking the derivative with respect to t is the same than taking the derivative with respect to x. How fast you drive does not depend on whether you drive in Zuerich time or Boston time.
b) In general the answer is $f'(x-a)$. One can see this also formally by taking the average rate of change $(f(x-a+h)-f(x-a))/h$, write it as $(f(t+h)-f(t))/h \to f'(t)/h$ then plug in again $t=x-a$ to get $f'(x-a)$.
c) The general is $cf'(cx+b)$. One can use what we did in class where we stated $d/dx f(cx) = cf'(x)$. Also this could be verified by looking at the average rate of change $[f(c(t+h)) - f(ct)]/h = c[f(ct+H)-f(ct)]/H$ with a scaled step $H = ch$. If you take the limit $H \to 0$ in the later expression you get $cf'(cx)$. If you take the limit $h \to 0$ in the first expression you get $d/dx f(cx)$.

Problem 7.3: Compute the following derivatives
a) $4\sin(3x) + 7\cos(9x)$
b) $\ln(x) + \ln(2x) + \ln(3x)$
c) $10e^{11x} + 8e^{20x} - 20e^{100x}$
d) $9x^2 + \frac{1}{x^7} + 2\cos(3x) + \ln(7x) - e^x$.
e) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$.

Solution:
a) $12\cos(3x) - 63\sin(9x)$
b) $\frac{1}{x} + \frac{1}{x} + \frac{1}{x} = 3/x$
c) $110e^{11x} + 160e^{20x} + 2000e^{100x}$.
d) $18x - \frac{7}{x^8} - 6\sin(3x) + 1/x - e^x$.
e) $1 - x + x^2 - x^3 + x^4$.

Problem 7.4: a) Compute the derivative of $f(x) = \pi^x$. You first might have to rewrite the function in a form which allows you to use rules you know.
b) What is the derivative of a^x in general if $a > 0$ is an arbitrary number?

Solution:
a) $\pi^x = e^{x \ln(\pi)}$. Now use the derivative rule to get $\ln(\pi)e^{x \ln(\pi)}$.
b) Do the same thing with $e^{x \ln(a)}$ to get the derivative $\ln(a)e^{x \ln(a)} = \ln(a)a^x$. If you want to learn the rule $d/dx a^x = \ln(a)a^x$, be my guest. Learning some more rules makes you a bit faster. The trade-off is that you have to remember more. This is a general phenomenon in computer science. With the use of more memory, one can in general accelerate processes. You gain speed but waste memory. There is no free lunch!

Problem 7.5: a) Compute the derivative of $f(x) = \sqrt{3x+5}$ from the rules you know.
b) In order to appreciate what we have achieved, compute the limit

$$\lim_{h \to 0} [f(x+h) - f(x)]/h \; .$$

for the function $f(x) = \sqrt{3x+5}$ the old way as in PSet 6.

Solution:
a) $\frac{3}{2\sqrt{3x+5}}$.
b) The "old way" is much more costly. The point of calculus is to have a tool avoiding such algebra acrobatics! By the way, the point of mathematics is **to solve a given problem in the most effective way!**. Effective can mean having less complexity or also to be more clear or intuitive. Note that sometimes, we want you to do something in a certain way. Like in this problem, where we forced you to solve the problem taking limits. Its like asking you to do pullups with weights attached to your belt. There is no practical necessity to do so, it just makes you stronger.

$$\sqrt{3(x+h)+5} - \sqrt{3x+5} = (\sqrt{3(x+h)+5} - \sqrt{3x+5}) \frac{(\sqrt{3(x+h)+5} + \sqrt{3x+5})}{(\sqrt{3(x+h)+5} + \sqrt{3x+5})}$$

Simplifies to

$$[3(x+h)+5 - (3x+5)]/[(\sqrt{3(x+h)+5} + \sqrt{3x+5})]$$

It can be divided by h go get $3/[(\sqrt{3(x+h)+5} + \sqrt{3x+5})]$. In the limit $h \to 0$ we get $3/[2\sqrt{3x+5}]$.

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