INTRODUCTION TO CALCULUS

MATH 1A

Unit 6: Derivative

6.1. The **derivative** of a function $f(x)$ at a point x is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists. It is the **instantaneous rate of change** we introduced earlier: it is the slope of the tangent at the point x.

6.2. Let us look at the function $f(x) = x^2$. We have

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{2hx + h^2}{h} .$$

For $h \neq 0$, we can divide by h and equate this to $2x + h$. We can now take the limit $h \to 0$ and see that $f'(x) = 2x$.

6.3. We will derive in class that in general, the function $f(x) = x^n$ leads to

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h} = \frac{nhx + \dots}{h}$$

which similarly as before simplifies to $nx^{n-1} + h(R(x))$, where $R(x)$ is a polynomial. Now again, for $h \to 0$, have

$$\boxed{\tfrac{d}{dx} x^n = nx^{n-1}}$$

6.4. For the **exponential function** $f(x) = e^x$, one can also compute the derivative

$$\frac{e^{x+h} - e^x}{h} = \frac{[e^x e^h - e^x]}{h} = e^x \frac{e^h - 1}{h} .$$

Now we can see that the limit $(e^h - 1)/h$ goes to 1 as $h \to 0$. Proving this depends on how the exponential function is defined. A calculator for example implements the exponential function as $e^h = 1 + h + h^2/2 + h^3/6 + \cdots$ from which you can see that $e^h - 1$ is divisible by h. Taking the limit $h \to 0$ gives then 1.

$$\boxed{\tfrac{d}{dx} e^x = e^x}$$

Homework

This homework is due on Monday 2/05/2024.

Problem 6.1: a) Find the limit $\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$ at the point $x = 2$.
b) Do the same with the function $x^3 - x$ instead of x^3.

Solution:
a) The expression $(x+h)^3 - x^3$ simplifies to $3x^2h + 3xh^2 + h^3$. Dividing by h gives $3x^2 + 3xh + h^2$. Now we can plug in $h = 0$ and get the answer. Evaluated at $x = 2$ we have 12.
b) We can reuse the computation from a) and see that the answer is $3x^2 - 1 = 11$.

Problem 6.2: a) Use the definition of the derivative to compute the derivative of $(x+1)/(x-2)$ at $x = 0$.
b) Compute the limit $\lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$.

Solution:
a) $(x+h+1)/(x+h-2) - (x+1)/(x-2)$ simplifies to $-3h/((x-2)(x+h-2))$. Dividing by h gives $-3/((x-2)(x+h-2))$ which now in the limit $h \to 0$ gives $-3/(x-2)^2$.
b) This needs a trick. You need to multiply both sides with $\sqrt{x+h} + \sqrt{x}$ and simplify to get $h/(\sqrt{4+h} + \sqrt{4})$. Dividing by h and setting $h = 0$ gives $1/(2\sqrt{4}) = 1/4$.

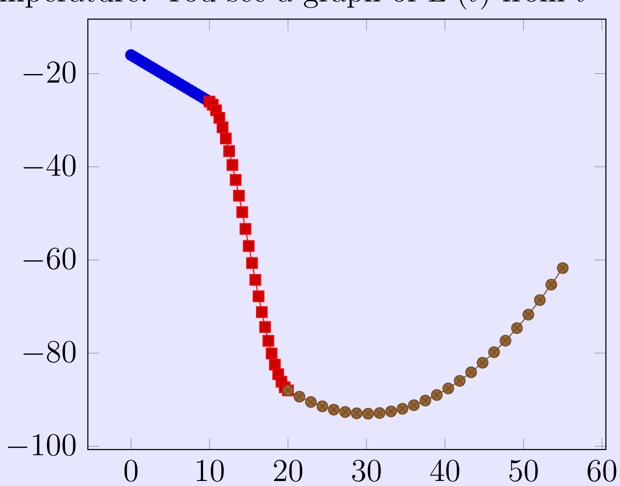
Problem 6.3: Give in each case a function with the property or explain why it does not exist:
a) f is continuous at $x = 3$ but not differentiable at $x = 3$.
b) f is continuous at $x = 3$ and differentiable at $x = 3$.
c) f is differentiable at $x = 3$ but not continuous at $x = 3$.

Solution:
a) $f(x) = |x - 3|$.
b) $f(x) = x^3$. c) This does not work. Differentiability implies that the function is continuous.

Problem 6.4: Find the limit $f'(x) = \lim_{h \to 0}[f(x+h) - f(x)]/h$. for $f(x) = 1/x$.

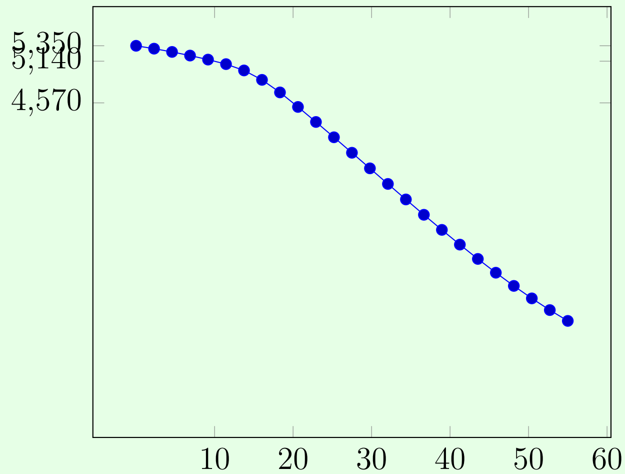
Solution:
Put the expression in one fraction. This gives $-h/(hx+x^2)$. Dividing by h gives $-1/(hx+x^2)$. Setting $h \to 0$ gives $-1/x^2$.

Problem 6.5: In this QRD problem we want to see what f' tells about f. The water level $L(t)$ of the **Aral sea** dropped between 1960 and 1970 by 21 cm/year, then from 1970-1980, it decreased by 57 cm per year. Afterwards until 2015 (t=50) the drop in water level started accelerating, due to positive feedback between evaporation and Sea Surface Temperature. You see a graph of $L'(t)$ from $t = 0$ to $t = 55$.



Assume that the water level was 54 m in 1960 (t=0), draw a qualitative picture on how $L(t)$ looks like.

Solution:
We know $L(0) = 5350$. L is always decreasing since L' is negative. It is concave down from $t = 0$ to $t = 10$, then more concave from $t = 10$ to $t = 30$ For $t > 30$ it is concave up (but still decreasing!). These are the essential features of the graph of L. Moreover, since the average water of water loss in the first decade was 21 cm/year, the water level dropped by 210 cm in the first decade, so it was $5350 - 210 = 5140$ cm at $t = 10$. In the second decade, the Aral Sea lost an average of 57 cm/year, so by $t = 20$, the water level was $5140 - 570 = 4570$. So, here is a possible graph of L:



Because L' is so negative for most of the time, the concavity change is a little difficult to see.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024