

# INTRODUCTION TO CALCULUS

MATH 1A

## Unit 5: Continuity

**5.1.** A function  $f$  is called **continuous** at a point  $x_0$  if a value  $f(x_0)$  can be found such that  $f(x) \rightarrow f(x_0)$  for  $x \rightarrow x_0$ . A function  $f$  is **continuous on the interval**  $[a, b]$  if it is continuous for every point  $x$  in the interval  $[a, b]$ . This means intuitively, we can **draw the graph of the function without lifting the pencil**. In  $(a, b)$ , the limit needs to exist both from the right and from the left. Continuity means that small changes in  $x$  results in small changes of  $f(x)$ . Any polynomial like  $x^3 + 2x - 4$  or trig functions like  $\cos(x)$ ,  $\sin(x)$  or exponential functions  $\exp(x)$  are continuous.

### Rules:

- a) If  $f$  and  $g$  are continuous, then  $f + g$  is continuous.
- b) If  $f$  and  $g$  are continuous, then  $f * g$  is continuous.
- c) If  $f$  and  $g$  are continuous and if  $g \neq 0$  everywhere, then  $f/g$  is continuous.
- d) If  $f$  and  $g$  are continuous, then  $f \circ g(x) = f(g(x))$  is continuous.

**5.2.** The **squeeze theorem** is a tool to check continuity at a point  $a$ .

If  $g(x) \leq f(x) \leq h(x)$  for functions  $g, h$  continuous at  $a$  and  $g(a) = h(a) = b$ , then  $f$  is continuous at  $a$  and  $f(a) = b$ .

The reason is that  $|h(x) - g(x)| \rightarrow 0$  and  $g(x) \rightarrow b$  and  $|f(x) - b| \leq |h(x) - g(x)| + |g(x) - b| \leq |h(x) - g(x)| + |g(x) - b| \rightarrow 0$  so that also  $f(x) \rightarrow b$ .

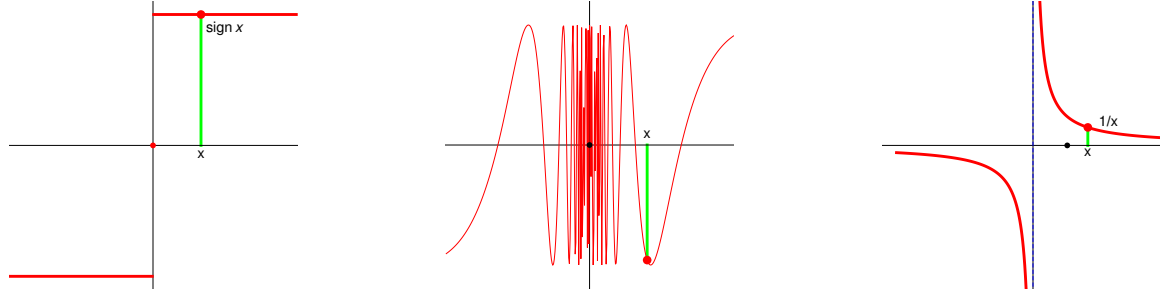
**5.3.** The function  $f(x) = 1/x$  is continuous except at  $x = 0$ . There is **pole discontinuity** at  $x = 0$ . The graph has a **vertical asymptote**.

**5.4.** The logarithm function  $f(x) = \ln|x|$  is continuous for all  $x \neq 0$ . It can not be fixed  $x = 0$  because  $f(x) \rightarrow -\infty$  for  $|x| \rightarrow 0$ .

**5.5.** The **co-secant function**  $\csc(x) = 1/\sin(x)$  is not continuous at  $x = 0, x = \pi$  and any multiple of  $\pi$ .

**5.6.** The function  $f(x) = \sin(\pi/x)$  is continuous everywhere except at  $x = 0$ . It fails continuity because of **oscillation**. We can approach  $x = 0$  in ways that  $f(x_n) = 1$  and such that  $f(z_n) = -1$ . Just pick  $x_n = 2/(4k + 1)$  or  $z_n = 2/(4k - 1)$ .

**5.7.** There are three major reasons, why a function can be not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity.



HOMEWORK, DUE FRIDAY 2/2/2024

**Problem 5.1:** a) Define  $f(x) = x^2 \cos(1/x)$  for  $x \neq 0$  and  $f(x) = 0$  for  $x = 0$ . Use the squeeze theorem to see that this function is continuous everywhere.  
b) Define  $f(x) = \cos(1/x)$  for  $x \neq 0$  and  $f(x) = 1$  for  $x = 0$ . Verify that this function is not continuous at 0.

**Problem 5.2:** The number of users (in millions) of a social network is modeled as a function that is linear  $U(t) = at + b$  for  $t \geq 5$  and exponential  $U(t) = ce^{kt}$  for  $t \in [0, 5]$ . Assume that we have 2 million users initially  $U(0) = 2$ . Data fitting leads to  $a = 220$  and  $b = 380$  for the linear growth. Determine  $k$  so that the function is continuous at 5.

**Problem 5.3:** a) The function  $f(x) = (e^{2x} - 1)/(e^x - 1)$  is not defined at  $x = 0$ . Can you find a value  $f(0) = b$  so that the function is continuous everywhere?  
b) The function  $f(x) = (x^3 + 2x^2 - 2x - 1)/(x - 1)$  is not defined at  $x = 1$ . Can you find a value  $f(1) = b$  so that with this postulate the function is continuous?

**Problem 5.4:** Which of the following functions are continuous everywhere?

- $f(x) = \text{sign}(x) + \sin(1/x)$
- $f(x) = x \text{sign}(x) + \sin(1/x)$
- $f(x) = \text{sign}(x) + x \sin(1/x)$
- $f(x) = x \text{sign}(x) + x \sin(1/x)$

**Problem 5.5:** Which functions can be made continuous by “fixing broken places” (assign a value to an initially not defined point)? If the function can be fixed show how.

- $\text{sinc}(5x) + \sin(x)/(2 + \sin^2(x)) + (x^3 - 1)/(x - 1)$ , b)  $\sin(\tan(x))$
- $\tan(\sin(x)) + \frac{x^2+5x+x^4}{x-3}$
- $\tan(2 \sin(x)) + \frac{x^2+5x+x^4}{x-3}$