INTRODUCTION TO CALCULUS

MATH 1A

Unit 4: Limits

4.1. We write $x \to a$ to indicate that x **approaches** a. It can approach **from the left** $x \to a^-$ or **from the right** $x \to a^+$. Write $\lim_{x\to a^-} f(x)$ for the **limit from the left** if it exists $\lim_{x\to a^+} f(x)$ for the **limit from the right** if it exists. If both limits exist and agree, we say that $\lim_{x\to a} f(x)$ exists. For $f(x) = \text{sinc}(x) = \sin(x)/x$ for example, the value at $a = 0$ is not defined. To investigate the limit from the right, evaluate $\text{sinc}(0.01) = \sin(0.01)/0.01 = 0.999983$ and $\text{sinc}(0.001) = 0.9999998333$. It looks as if the limit from the right is 1. The function is even, meaning $f(-x) = f(x)$ so that also the limit from the left appears to be 1. The function $\text{sinc}(x) = \sin(x)/x$, used **signal processing** will be discussed more in class.

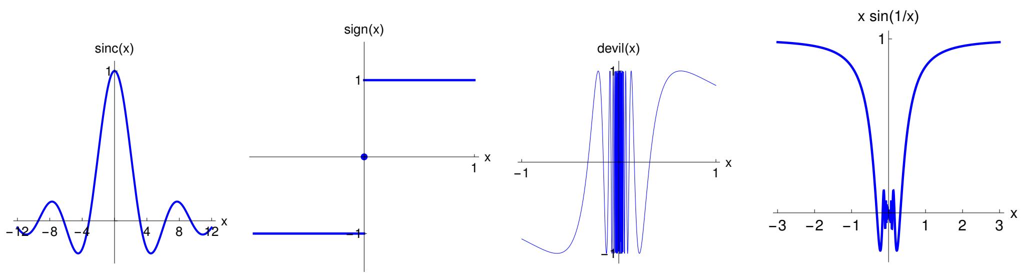


FIGURE 1. The sinc function, the sign function, the devil function $f(x) = \sin(1/x)$ and the **tamed devil function** $f(x) = x\sin(x/x)$.

4.2. $f(x) = \text{sign}(x)$ is defined to be 1 of $x > 0$ and 0 for $x = 0$ and equal to -1 for $x < 0$. In this case, both left and right limits do not exist. Can you modify the function so that the left limit exists and the right limit does not exist?

4.3. For $f(x) = \frac{1-\cos(x)}{x^2}$, it is not so clear what happens at $a = 0$. We can not plug in $x = 0$ because then we divide 0 over 0. But we can evaluate the function for x values close to $x = 0$ and see what happens. Lets see: $\frac{1-\cos(0.1)}{0.1^2} = 0.499583$ $\frac{1-\cos(0.01)}{0.01^2} = 0.499996$. What actually happens is that $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = 1/2$. We will learn how to check that later in the course.

4.4. $f(x) = 1/x^2$ is not defined at $a = 0$. This function has a **pole** at $x = 0$ or a "vertical asymptote".

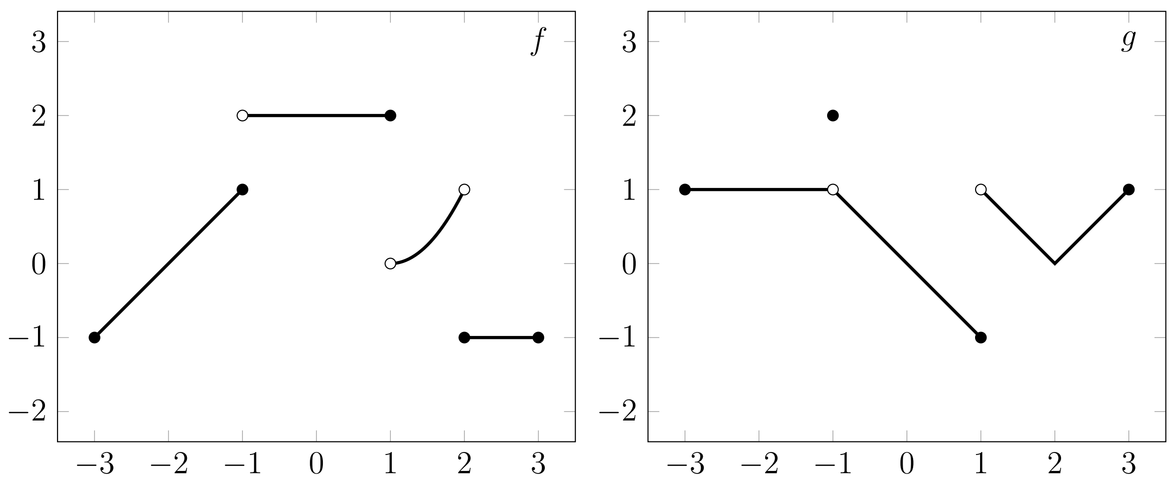
4.5. For the function $f(x) = \sin(1/x)x$ for $x \neq 0$ and $f(0) = 0$ we have $|f(x)| \leq |x|$. The function $f(x)$ converges to 0 for $x \to 0$.

4.6. The function $f(x) = x \log_{10}|x|$ is not defined at $x = 0$. But we can look at small numbers $x > 0$ and small numbers $x < 0$ to investigate the values. We check for example $f(10^{-k}) = \frac{-k}{10^k}$. and $f(-10^{-k}) = \frac{k}{10^k}$. Can you see the limit?

Homework, due Wednesday 1/31/2024

Problem 4.1: a) We do not know the value of $f(x) = (x^4 - 1)/(x - 1)$ at $x = 1$. Nevertheless there is a natural value which can attach to at $x = 1$. Find this value.
b) Use a calculator to find the limit $\lim_{x \to 1} \frac{2^x - 2}{x-1}$. Make a convincing statement why the left or right limits exist or not.

Problem 4.2: For both f and g below, find the left and right limit at the points $-2, -1, 0, 1, 2$. At $x = 3$ only the left limits, at $x = -3$ only the right limits are needed.



Problem 4.3: Investigate the function $f(x) = \sin(\pi/x)$ at the point $x = 0$. a) First draw the graph of $\sin(\pi/x)$ on the interval $[-2, 2]$. b) Especially evaluate the function at $-2, -1, -1/2, -1/3, -1/4, 1/4, 1/3, 1/2, 1, 2$ and mark them on your graph.

Problem 4.4: a) Draw the graph of a function such that $\lim_{x \to 0^+} f(x) = 1, \lim_{x \to 0^-} f(x) = 2$ and for which $f(0) = 0$. Use the drawing conventions you observe from the previous problem.
b) Find a concrete example of a function for which the limit $\lim_{x \to a}$ does not exist at every point but for which the function is defined at every point. If you are stuck, grab some salt and pepper and season the paper with it. Then eat it.

Problem 4.5: Explore the function $f(x) = x^x$ at $x = 0$. Make experiments. Can you say something about the limit from the right or limit from the left?