INTRODUCTION TO CALCULUS

MATH 1A

Unit 4: Limits

4.1. We write $x \to a$ to indicate that x **approaches** a. It can approach **from the left** $x \to a^-$ or **from the right** $x \to a^+$. Write $\lim_{x \to a^-} f(x)$ for the **limit from the left** if it exists $\lim_{x \to a^+} f(x)$ for the **limit from the right** if it exists. If both limits exist and agree, we say that $\lim_{x \to a} f(x)$ exists. For $f(x) = \text{sinc}(x) = \sin(x)/x$ for example, the value at $a = 0$ is not defined. To investigate the limit from the right, evaluate $\text{sinc}(0.01) = \sin(0.01)/0.01 = 0.999983$ and $\text{sinc}(0.001) = 0.9999998333$. It looks as if the limit from the right is 1. The function is even, meaning $f(-x) = f(x)$ so that also the limit from the left appears to be 1. The function $\text{sinc}(x) = \sin(x)/x$, used **signal processing** will be discussed more in class.

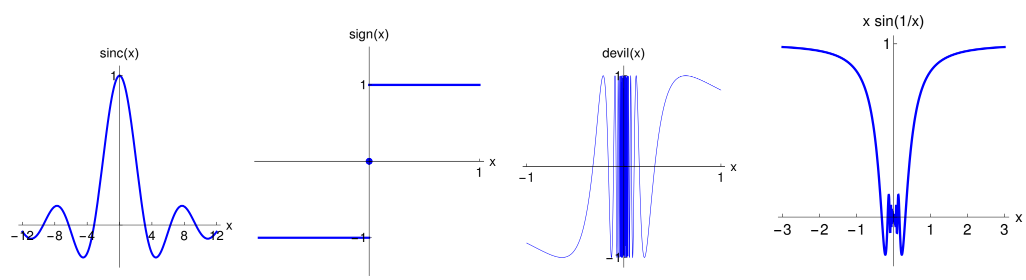


FIGURE 1. The sinc function, the sign function, the devil function $f(x) = \sin(1/x)$ and the **tamed devil function** $f(x) = x\sin(x/x)$.

4.2. $f(x) = \text{sign}(x)$ is defined to be 1 of $x > 0$ and 0 for $x = 0$ and equal to -1 for $x < 0$. In this case, both left and right limits do not exist. Can you modify the function so that the left limit exists and the right limit does not exist?

4.3. For $f(x) = \frac{1-\cos(x)}{x^2}$, it is not so clear what happens at $a = 0$. We can not plug in $x = 0$ because then we divide 0 over 0. But we can evaluate the function for x values close to $x = 0$ and see what happens. Lets see: $\frac{1-\cos(0.1)}{0.1^2} = 0.499583$ $\frac{1-\cos(0.01)}{0.01^2} = 0.499996$. What actually happens is that $\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = 1/2$. We will learn how to check that later in the course.

4.4. $f(x) = 1/x^2$ is not defined at $a = 0$. This function has a **pole** at $x = 0$ or a "vertical asymptote".

4.5. For the function $f(x) = \sin(1/x)x$ for $x \neq 0$ and $f(0) = 0$ we have $|f(x)| \leq |x|$. The function $f(x)$ converges to 0 for $x \to 0$.

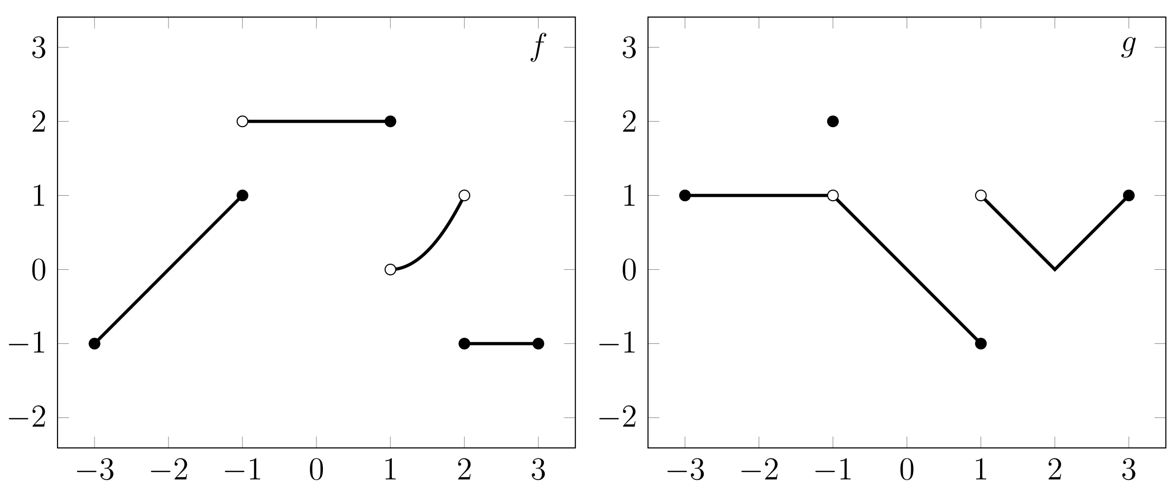
4.6. The function $f(x) = x\log_{10}|x|$ is not defined at $x = 0$. But we can look at small numbers $x > 0$ and small numbers $x < 0$ to investigate the values. We check for example $f(10^{-k}) = \frac{-k}{10^k}$. and $f(-10^{-k}) = \frac{k}{10^k}$. Can you see the limit?

Homework, due Wednesday 1/31/2024

Problem 4.1: a) We do not know the value of $f(x) = (x^4 - 1)/(x - 1)$ at $x = 1$. Nevertheless there is a natural value which can attach to at $x = 1$. Find this value. b) Use a calculator to find the limit $\lim_{x\to 1} \frac{2^x - 2}{x-1}$. Make a convincing statement why the left or right limits exist or not.

Solution:
a) First factor $x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$. Divide by $x - 1$ gives $(x+1)(x^2+1)$. Now we can evaluate at $x = 1$ and get the value $\boxed{4}$. b) Take a number close to 1 like 1.0001. Now compute $(2^{1.0001} - 2)/(1.0001 - 1) = 1.386$. Also check a number to the left like 0.9999 and get again a value 1.386. (It turns out to be $2\ln(2) = \ln(4)$ as we will see later in the course). The argument can be made more convincing by taking numbers even closer to 1. Also convincing is to plot the graph of the function and see that there appears to be a definite value at 1.

Problem 4.2: For both f and g below, find the left and right limit at the points $-2, -1, 0, 1, 2$. At $x = 3$ only the left limits, at $x = -3$ only the right limits are needed.



Solution:
Lets look at f first:

x value	left limit	right limit
-3	-	-1
-2	0	0
-1	1	2
0	2	2
1	2	0
2	1	-1
3	-1	-

Now look at g

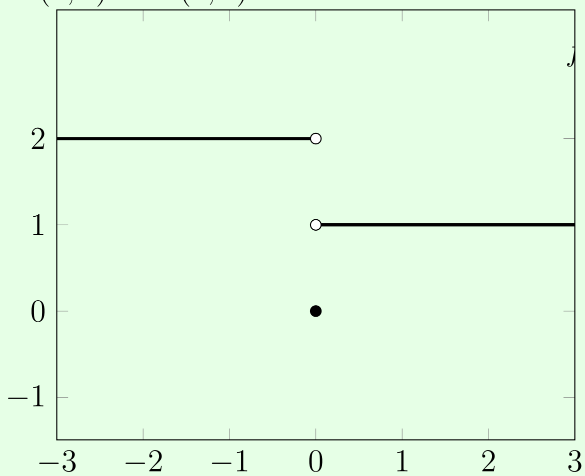
x value	left limit	right limit
-3	-	1
-2	1	1
-1	1	1
0	0	0
1	-1	1
2	0	0
3	1	-

Problem 4.3: Investigate the function $f(x) = \sin(\pi/x)$ at the point $x = 0$. a) First draw the graph of $\sin(\pi/x)$ on the interval $[-2, 2]$. b) Especially evaluate the function at $-2, -1, -1/2, -1/3, -1/4, 1/4, 1/3, 1/2, 1, 2$ and mark them on your graph.

Solution:
a) There is an oscillatory singularity. There is no limit at 0. b) The values are all zero at $-1, -1/2, -1/3, -1/4, 1/4, 1/3, 1/2, 1$ and 1 $x = 2$ and -1 at -2.

Problem 4.4: a) Draw the graph of a function such that $\lim_{x\to0^+} f(x) = 1, \lim_{x\to0^-} f(x) = 2$ and for which $f(0) = 0$. Use the drawing conventions you observe from the previous problem.
b) Find a concrete example of a function for which the limit $\lim_{x\to a}$ does not exist at every point but for which the function is defined at every point. If you are stuck, grab some salt and pepper and season the paper with it. Then eat it.

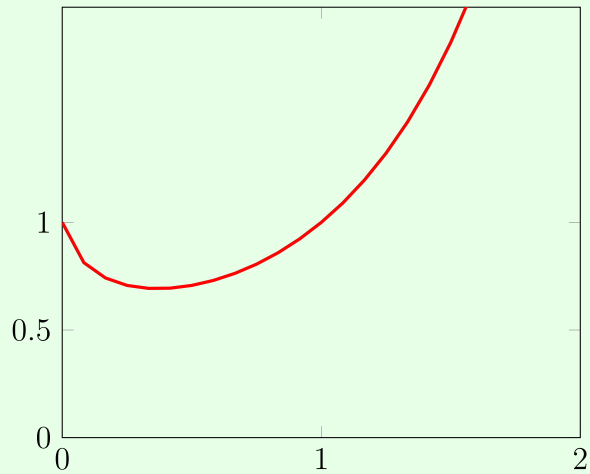
Solution:
a) Draw lines 1 on $x > 0$ and 2 for $x < 0$ and drop a filled bullet at 0 and an empty bullet on $(0, 2)$ and $(0, 1)$.



b) Google "salt and pepper function" or "salt and pepper in calculus". The salt and pepper function is defined to be the function which satisfies $f(x) = 1$ at rational values $x = p/q$ and 0 at rational values like $x = \sqrt{2}$. The limit does not exist at any point because every rational number has irrational numbers arbitrary close and every irrational number has rational numbers arbitrarily close. We took this example because it is easy to state but hard to find. If you were asked to find a function which has the property that the limit does not exist at every point, you would be stumped.

Problem 4.5: Explore the function $f(x) = x^x$ at $x = 0$. Make experiments. Can you say something about the limit from the right or limit from the left?

Solution:
The function is not defined for $x < 0$. (The calculator does not draw a graph for $x < 0$). For $x > 0$ it converges to 1. Take for example $0.0001^{0.0001}$ and get 0.999079. One can take small values to experiment. If one would like to analyze this, look at $e^{x \ln(x)}$ and note that $x \ln(x)$ goes to 0 for $x \to 0$. Take for example $x = 1/e^{100}$ which gives $100/e^{100}$ which is very small.



It is very interesting that the function $f(x) = |x|^{|x|}$ with the understanding $f(0) = 1$ is actually a continuous function!

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024