INTRODUCTION TO CALCULUS

MATH 1A

Unit 3: Rate of Change

Average rate of change

3.1. The **average rate of change** of a function f on the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a} .$$

The **instantaneous rate of change** at a point x is for now informally defined as the **slope of graph function** at x. The name "average rate of change" will be justified later as it will be identified with the average of all derivatives $f'(x)$ on the interval $[a, b]$. Once we will see the concept of limit, we will identify the **instantaneous rate of change** as a **limit** $[f(x+h) - f(x)]/h$ when h goes to zero. We will get to soon. It is a bit puzzling as when $h = 0$, we have an indeterminate form $0/0$. For this current lecture, it we work with the **slope of the tangent** at x.

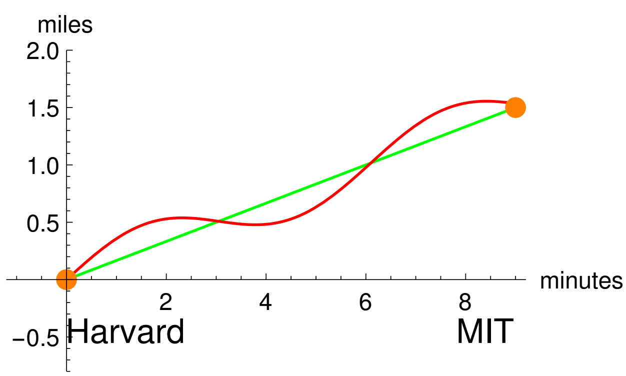


FIGURE 1. A trip can be quantified by the function giving the distance from Harvard as a function of time.

3.2. The average rate of change also makes sense if the function is known only for a few data points. Assume you run the $2km$ from **Harvard to MIT** along the Charles and you clock it with 12 minutes. Then your average speed is $2/12 = 1/6$ km per minute. The instantaneous rate of change varies over the run. You might run slower before the Boston university bridge but then accelerate near the de Wolfe Boathouse, where it goes down. If we measured the progress every second a few times, we would see also that there are variations related to small variations related to getting off and landing on the floor. If you have a smart watch, it measures average rates of changes by looking up your location using **GPS satellites** every once in a while. The smart watch would talk to you during the run and inform you that you ran with $6min/km$.

Instantaneous rate of change

3.3. The **instantaneous rate of change** will later be defined as a limit of average rates of changes when the interval $[x, x+h]$ gets smaller and smaller, but it can also be understood intuitively and geometrically as the **slope of the tangent** at the graph. If you look at the function $f(x) = x^2$, the slope at a point x is $2x$. Lets look at the average rate of change on $[0, h]$ for this function. It is $[f(x+h) - f(x)]/h = [(x+h)^2 - x^2]/h$. If we foil out this expression and simplify, it becomes $[x^2 + 2hx + h^2 - x^2]/h = 2x + h$. One can see that if h goes to zero, the limit $2x$ appears. This value $2x$ what we will call the derivative of f at the point x.

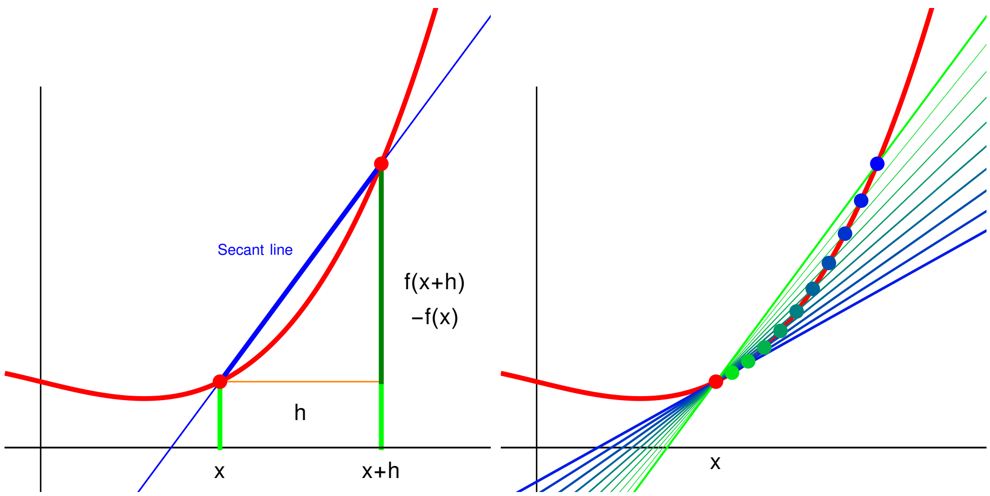


FIGURE 2. The average rate of change is the slope of the secant connecting two points on the graph. The instantaneous rate of change is a limiting value when the intervals get smaller and smaller. It is for now the slope of the tangent at x.

3.4. Historically, the notion of derivative needed time to develop. One of the first, who investigated the notion seriously was **Zeno of Elea** who was born around 490 BC, just around the time, when Pythagoras (570-495 BC) died. Already Aristotle objected to the paradoxa that Zeno relied on the false supposition that time is composed of indivisible "nows" or "instants". Since we do not know how space and time looks on the **Planck scale** The question of Zeno remains of interest today. [1]

3.5. The nice thing about the average rate of change is that it does not use any notion of limit.

[1] See the book "Zeno's paradox'" (2008) by Joseph Mazur or watch "Ant man and the Wasp"

This homework is due on Monday 2/29, 2024.

Problem 3.1: The 10 meter marks of **Usain Bolt'** during his record run during the Beijing 2008 Olympics are:

Distance	10	20	30	40	50	60	70	80	90	100
Time	1.85	2.87	3.78	4.65	5.50	6.32	7.14	7.96	8.79	9.69

a) What is Bolt's average speed over the entire range?
b) What is Bolt's average speed over the first 50 meters?
c) What is Bolt's average speed over the last 50 meters?
d) Is the average of b) and c) equal to the result in a)?
e) Over which of the 10 meter intervals is the average speed highest?

Solution:
a) The average speed is $\frac{\text{distance}}{\text{time}} = \frac{100 \text{ m}}{9.69 \text{ s}} \approx \boxed{10.32 \text{ meters per second}}$
b) We have $\frac{50 \text{ m}}{5.50 \text{ s}} \approx \boxed{9.09 \text{ meters per second}}$
c) The last 50 meters of the race took Bolt $9.69 - 5.50 = 4.19$ seconds, so his average speed was $\frac{50 \text{ m}}{4.19\text{s}} \approx \boxed{11.93 \text{ meters per second}}$. d) No. Averaging the answers gives $\approx \frac{9.09+11.93}{2} =$ 10.51 meters per second, which is not the same.
e) The average speed is $\frac{\text{distance}}{\text{time}}$. The average speed with 10 meter spacing is highest when the time used for that interval is lowest. The times in each interval are

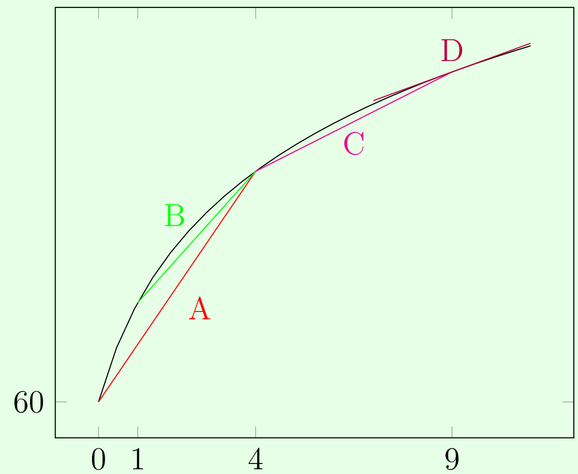
Interval (meters)	0 to 10	10 to 20	20 to 30	30 to 40	40 to 50
Time (seconds)	1.85	1.02	0.91	0.87	0.85

Interval (meters)	50 to 60	60 to 70	70 to 80	80 to 90	90 to 100
Time (seconds)	0.82	0.82	0.82	0.83	0.90

The average speed is tied for highest among the intervals $\boxed{\text{50 to 60 meters}}$, $\boxed{\text{60 to 70 meters}}$, and $\boxed{\text{70 to 80 meters}}$. Because the graph is concave down, the slopes get smaller as we move from left to right. From the picture we see $\boxed{D < C < B < A}$.

Problem 3.2: The population of **Pandora** between 2000 and 2024 is modeled by the function $P(t) = 60 + 3\log_2(t+1)$, giving the number of millions of Na'vi people on the planet t years after 2000.
a) What is the average rate of change of the population between $t = 1$ and $t = 7$?
b) Sketch the graph of $P(t)$. From this graph, without computing the numbers order the following quantities in increasing numbers:
A) The average rate of change between $t = 0$ and $t = 4$ B) The average rate of change between $t = 1$ and $t = 4$ C) The average rate of change between $t = 4$ and $t = 9$ D) The instantaneous rate of change at $t = 9$.

Solution:
a) $(\log_2(8) - \log_2(2))/(7-1) = (3-1)/6 = 1/3$.
b)



Problem 3.3: a) For a given function $f(x)$ denote with $Df(x) = f(x+1) - f(x)$ the average rate of change between x and $x+1$. Compute this for the function $f(x) = x(x-1)$.
b) Now change the size of the interval and let $Df(x) = (f(x+h) - f(x))/h$ denote the average rate of change between x and $x+h$. Compute this for the function $f(x) = x(x-h)$.
c) Can you see the instantaneous rate of change of $f(x) = x^2$ at a point x?

Solution:
a) We have $f(x+1) = (x+1)x$ and $f(x+1) - f(x) = (x+1)x - x(x-1) = x(x+1-(x-1)) = \boxed{2x}$.
b) We have $f(x+h) = (x+h)x$ and $f(x+h) - f(x) = (x+h)x - x(x-h) = x(x+h-(x-h)) = 2xh$.
Dividing by h gives $\boxed{2x}$. c) Since the result always was $2x$ independent of h, we also have an instantaneous rate of change $2x$ in the limit. We talk about the limit later. We can also see it as such (also seen in class already) $f(x+h) - f(x) = (x+h)^2 - x^2 = 2xh + h^2$ can be divided by h. Therefore $[f(x+h) - f(x)]/h$ is $2x + h$. We can now safely look at the limit when $h \to 0$ by just setting $h = 0$. The result is again $\boxed{2x}$.

Problem 3.4: Repeat what you did in problem 3.3 a),b) for the function $f(x) = 2^x$. We can not yet compute the limit $h \to 0$ for the expression you get in c) but just compute it for a very small number like $h = 1/1000$.

Solution:
a) We have $f(x+1) = 2^{x+1} + 22^x$ and $f(x+1) - f(x) = 2^x(2-1) = 2^x$.
b) We similarly compute $f(x+h) = 2^{x+h} = 2^x 2^h$ and $f(x+h) - f(x) = 2^x(2^h - 1)$ c) The instantaneous rate of change is approximated by $2^x(2^h-1)/h$. we will see later that the limit $(2^h-1)/h$ for $h \to 0$ is $\ln(2) = 0.693...$. For $h = 1/1000$ we get $0.6933...2^x$.

Problem 3.5: This is an exploratory question which you definitely need to discuss:
a) Assume you have a function which has the property that the average rate of change between x and $x+1$ is 0 for all x? Does it mean that $f(x)$ is constant? Or can you find an example of a non-constant function where this happens?
b) Assume you have a function which has the property that the instantaneous rate of change at every point exists and is 0. Does it mean that $f(x)$ is constant? If the answer is no, provide a non-constant function with that property.

Solution:
a) Any function which is 1-periodic works like $\sin(2\pi x)$. Such a function has the property that $f(x+1) - f(x) = 0$. But this function is not constant.
It is important that you can massage a function like $\sin(x)$ to become periodic with some given period. In general, the period (wave length) for $\sin(cx)$ is $2\pi/c$. If you take a function like $\sin(10x)$, it has a high wave length. A function $\sin(0.1x)$ has a small wave length.
b) If the function has slope 0 everywhere, then this means that the function can not increase. Assume the slope would not be zero at some point x and be $c > 0$ then $[f(x+h) - f(x)]/h > c/2$ for small enough h so that also the limiting slope would have the property that it is bigger than $c/2$. So, no, a function for which the slope is zero everywhere must be constant.

Some figures

3.6. Some figures:

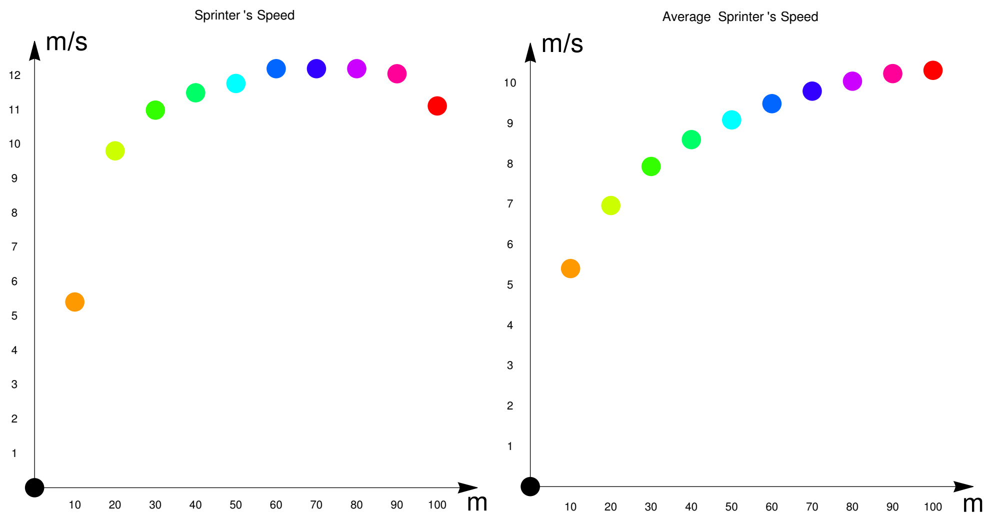


Figure 3. The first graph shows the average speed of Bolt on the intervals $[0, 10], [10, 20], \dots, [90, 100]$. This is the **rolling speed** and an approximation for the **instantaneous rate of change**. The second graph shows the **average speed** on the already traversed part $[0, x]$, where x is the number of meters. The last point shows the average rate of change of the run over the entire interval from 0 to 100 m. A speed of 12 m/s corresponds to 43km/h. One year later, in Berlin in 2009, Bolt ran a 9.58 second 100 meter final.

Figure 4. Part of the renaissance fresco **"School of Athens"** by Raphael painted around 1510 shows **Zeno of Elea** to the very right. He runs away in terror after realizing that protection from arrows by following his mantra ("There is no motion because at every moment, the arrow is fixed!"), does not seem to work.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024