INTRODUCTION TO CALCULUS

MATH 1A

Unit 2: Functions

Lecture

2.1. A **function** is a rule which assigns to a real number a new real number. The **domain** of f consists of the points where f is defined and a **codomain** B a set of numbers in which f is mapped to. The **range** is $f(A)$. A function $g(x) = 1/x$ for example can not be evaluated at 0 so that the domain must exclude the point 0. Its range is also $\mathbb{R} \setminus \{0\}$, the set of real numbers without 0. The **inverse** of a function f is a function g such that $g(f(x)) = x$. The function $g(x) = \sqrt{x}$ for example is the inverse of the function $f(x) = x^2$ on its domain $\mathbb{R}^+ = [0, \infty)$. The function $f(x) = 1/x$ is its own inverse on $(-\infty, 0) \cup (0, \infty)$.

identity	x
linear	$3x + 1$
quadratic	x^2
cosine	$\cos(x)$
sine	$\sin(x)$

exponential	$e^x = \exp(x)$
logarithm	$\ln(x) = \log(x)$
square root	$\sqrt{x}$
absolute value	$\|x\|$
bell function	e^{-x^2}

We can build new functions by:

addition	$f(x) + g(x)$
multiplication	$f(x) \cdot g(x)$
division	$f(x)/g(x)$
scaling	$2f(x)$
translation	$f(x+1)$
composition	$f(g(x))$
inverting	$f^{-1}(x)$

Important classes of functions are

polynomials	$x^2 + 3x + 5$
rational functions	$\frac{x+1}{x^4+1}$
exponentials	e^x, b^x
logarithm	$\ln(x), \log_b(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arctan(x)$
roots	$\sqrt{x}, x^{1/3}$

2.2. The **graph** $\{(x, y) = (x, f(x))\ \}$ allows to **visualize** functions. We can "see a function", when we draw the graph. We will learn to graph in class.

Homework

This homework is due on Friday 1/26, 2024.

Problem 2.1: Model the height $h(t)$ of the sea level at time t in Boston, where the tidal range is 10ft. You can assume that the time between low tide and high tide 2π hours and at time $t = 0$, the sea level $h(t)$ is lowest and equal to 0. Give a concrete function $h(t)$ which does the job.

Problem 2.2: The annual mean atmospheric CO_2 concentration $C(x)$ is observed over time.
a) The concentration was measured to be 300 parts per million in 1974 and 400 parts per million in 2024. Find a linear function $C(x) = ax + b$ which models the situation.
b) Now model the situation as an exponential function $C(x) = ab^x$ so that it fits the data.

Problem 2.3: The graph of the **income distribution** of a country is plotted as a function $L(x)$ of x, the bottom fraction of the population, is called the **Lorenz curve**. For example, if the bottom 30 percent earn 5 percent of the country's income then $L(0.3) = 0.05$. The Lorenz curve is shown below.
a) The **Gini index** is defined to be twice the area between the graph of $f(x) = x$ and the graph of $L(x)$. Use the picture to estimate the Gini index of the US.
b) What would the Gini index be, if everybody would earn the same income? We call this a **perfectly equal income distribution**.
c) How large can the Gini index become theoretically? What would that mean? Draw the situation where the bottom 99 percent earn nothing at all and the top 1 percent earns all the income. This is close to the **perfectly unequal income distribution**.

Problem 2.4: The **20:20 ratio** for a country is defined as the ratio of the country's total income earned by the top 20 percent of the households divided by the country's total income earned by the bottom 20 percent. In other words, it is $L(0.8)/L(0.2)$.
a) Find the 20:20 ratio for the USA using the graph given above.
b) What is the 20:20 ratio for a **perfectly equal country**.
c) What is the 20:20 ratio for a **perfectly unequal country**.

Problem 2.5: a) Plot the function $x \sin(1/x)$. Feel free to use technology if you like. Describe some features of this graph.
b) What would you hear, if a membrane would swing like function $\sin(4000\sqrt{x})$ say from $x = 0$ to $x = 10$? Describe this in words, like "wailing firetruck sirene" or a melody for which the "pitch increases in time".

2.3. Here is the graph for Problem 3 and 4:

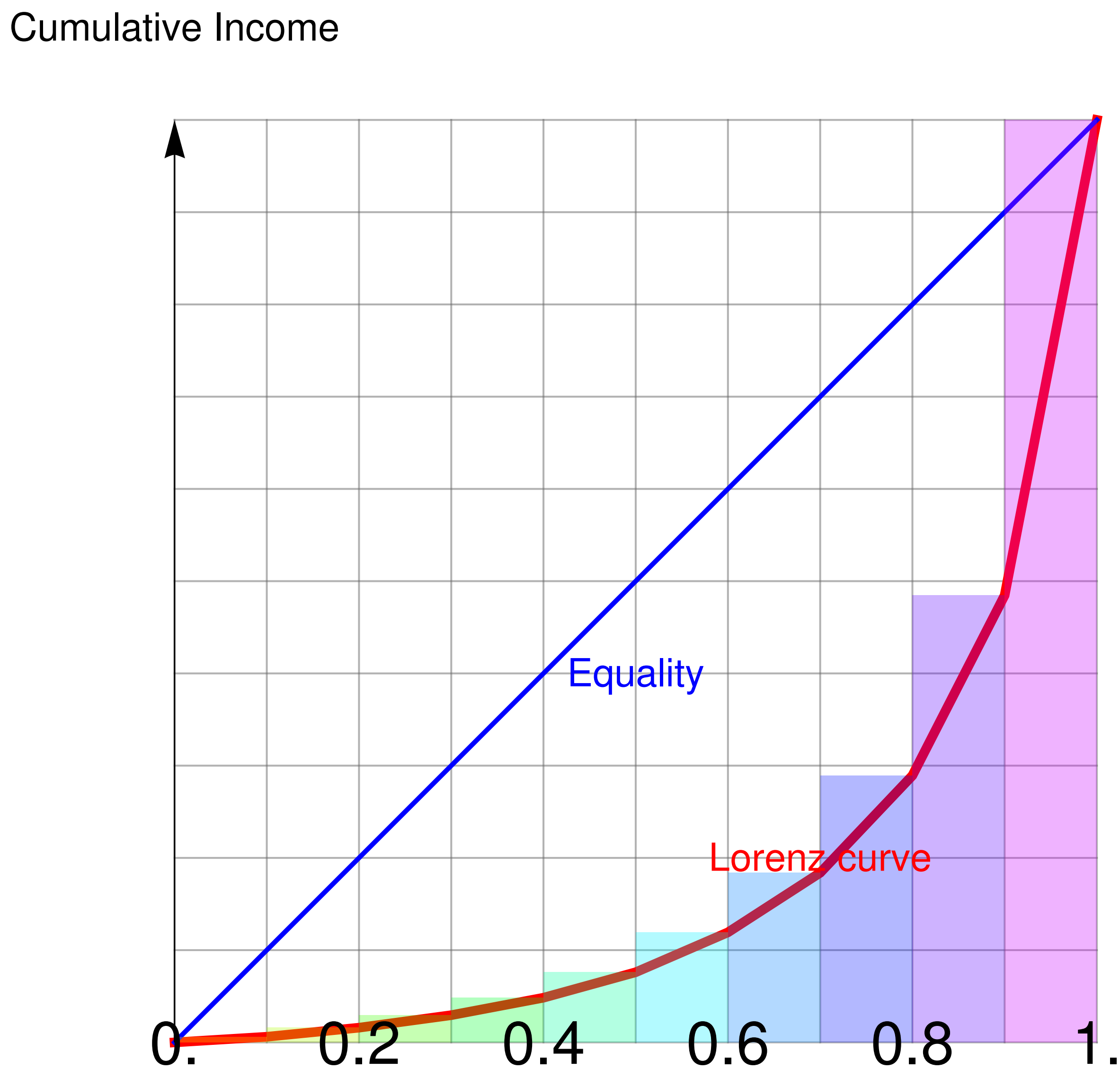

Figure 1. The Lorenz curve.

2.4. The income data points we worked with are:

0	161	260	358	500	730	1142	1707	2784	5161	13613

2.5. This means that the data points for the red curve are (if each square is considered a unit square): (these are the numbers you can work with and which agree with the graph shown above).

0	0.061	0.159	0.295	0.484	0.761	1.193	1.839	2.894	4.847	10.0

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