INTRODUCTION TO CALCULUS

MATH 1A

Unit 2: Functions

Lecture

2.1. A **function** is a rule which assigns to a real number a new real number. The **domain** of f consists of the points where f is defined and a **codomain** B a set of numbers in which f is mapped to. The **range** is $f(A)$. A function $g(x) = 1/x$ for example can not be evaluated at 0 so that the domain must exclude the point 0. Its range is also $\mathbb{R} \setminus \{0\}$, the set of real numbers without 0. The **inverse** of a function f is a function g such that $g(f(x)) = x$. The function $g(x) = \sqrt{x}$ for example is the inverse of the function $f(x) = x^2$ on its domain $\mathbb{R}^+ = [0, \infty)$. The function $f(x) = 1/x$ is its own inverse on $(-\infty, 0) \cup (0, \infty)$.

identity	x
linear	$3x + 1$
quadratic	x^2
cosine	$\cos(x)$
sine	$\sin(x)$

exponential	$e^x = \exp(x)$
logarithm	$\ln(x) = \log(x)$
square root	$\sqrt{x}$
absolute value	$\|x\|$
bell function	e^{-x^2}

We can build new functions by:

addition	$f(x) + g(x)$
multiplication	$f(x) \cdot g(x)$
division	$f(x)/g(x)$
scaling	$2f(x)$
translation	$f(x + 1)$
composition	$f(g(x))$
inverting	$f^{-1}(x)$

Important classes of functions are

polynomials	$x^2 + 3x + 5$
rational functions	$\frac{x+1}{x^4+1}$
exponentials	e^x, b^x
logarithm	$\ln(x), \log_b(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arctan(x)$
roots	$\sqrt{x}, x^{1/3}$

2.2. The **graph** $\{(x, y) = (x, f(x))\ \}$ allows to **visualize** functions. We can "see a function", when we draw the graph. We will learn to graph in class.

Homework

This homework is due on Friday 1/26, 2024.

Problem 2.1: Model the height $h(t)$ of the sea level at time t in Boston, where the tidal range is 10ft. You can assume that the time between low tide and high tide 2π hours and at time $t = 0$, the sea level $h(t)$ is lowest and equal to 0. Give a concrete function $h(t)$ which does the job.

Solution:
We want to take a trigonometric function which is periodic. In order to have 2π between high and low tide and periodicity, this means that the function should repeat after 4π. We need therefore to take $\sin(x/2)$ or $\cos(x/2)$. Since the minimum is at 0 we need to take $-\cos(x/2)$. Since the total range of amplitude is 10, we need to take $-5\cos(x/2)$. Since at $t = 0$ we get 0, we must move the graph up and get $5 - 5\cos(x/2)$. The answer is $\boxed{h(t) = 5 - 5\cos(x/2)}$.

Problem 2.2: The annual mean atmospheric CO_2 concentration $C(x)$ is observed over time.
a) The concentration was measured to be 300 parts per million in 1974 and 400 parts per million in 2024. Find a linear function $C(x) = ax + b$ which models the situation.
b) Now model the situation as an exponential function $C(x) = ab^x$ so that it fits the data.

Solution:
a) We know $C(1974) = 300$ and $C(2024) = 400$. This means that in 50 years we have an increase of 100. The slope a of the graph is therefore 2. Now fix the constant $1974*2+b = 300$ which gives $b = 300 - 2 * 1974 = -3648$. The solution is $\boxed{C(x) = 2x - 3648}$.
b) It is easier to translate the picture and take 1974 as the origin. Now we want $C(0) = ab^0 = 300$ which gives $a = 300$. Now take $C(50) = ab^{50} = 300b^{50} = 400$ which gives $b^{50} = 4/3$ and so $b = (4/3)^{1/50}$. The solution with 1974 as origin is $300(4/3)^{x/50}$. Now translate to get $C(x) = 300(4/3)^{(x-1974)/50}$.

Problem 2.3: The graph of the **income distribution** of a country is plotted as a function $L(x)$ of x, the bottom fraction of the population, is called the **Lorenz curve**. For example, if the bottom 30 percent earn 5 percent of the country's income then $L(0.3) = 0.05$. The Lorenz curve is shown below.
a) The **Gini index** is defined to be twice the area between the graph of $f(x) = x$ and the graph of $L(x)$. Use the picture to estimate the Gini index of the US.
b) What would the Gini index be, if everybody would earn the same income? We call this a **perfectly equal income distribution**.
c) How large can the Gini index become theoretically? What would that mean? Draw the situation where the bottom 99 percent earn nothing at all and the top 1 percent earns all the income. This is close to the **perfectly unequal income distribution**.

Solution:
a) There are 100 small squares are area 1/100 each. It depends on the counting but we can estimate about 32 squares. The area between the red and blue curve is about 0.32. The Gini index is $\boxed{0.64}$. (Anything between 0.55 and 0.75 should be ok). b) In that case the Lorenz curve would be $L(x) = 0$ so that the Gini index is $\boxed{0}$.
c) In the maximal case, the area is 1/2 and the Gini index would be $\boxed{1}$.

Problem 2.4: The **20:20 ratio** for a country is defined as the ratio of the country's total income earned by the top 20 percent of the households divided by the country's total income earned by the bottom 20 percent. In other words, it is $L(0.8)/L(0.2)$.
a) Find the 20:20 ratio for the USA using the graph given above.
b) What is the 20:20 ratio for a **perfectly equal country**.
c) What is the 20:20 ratio for a **perfectly unequal country**.

Solution:
a) It is $L(0.8)/L(0.2) = 2.894/0.159 = 18.2$
b) It is $0.8/0.2 = 4$
c) Infinite (or very large). One way to get a large number is to assume that $L(x) = x^{100}$ then $0.8^{100}/0.2^{100} = 4^{100}$ is very large .

Problem 2.5: a) Plot the function $x\sin(1/x)$. Feel free to use technology if you like. Describe some features of this graph.
b) What would you hear, if a membrane would swing like function $\sin(4000\sqrt{x})$ say from $x = 0$ to $x = 10$? Describe this in words, like "wailing firetruck sirene" or a melody for which the "pitch increases in time".

Solution:
a) You can do that in Desmos or in Mathematica. This is a tamed devil comb. It is continuous but not smooth. It has infinitely many oscillations near 0. The amplitudes go to zero as x goes to 0.
b) To hear the function, you have to either install mathematica or have remember the part of the lecture, where the relation of sound and function has been explained. $\sin(5000x)$ produces a constant frequency sound $\sin(7000x)$ is a higher pitch.
High frequency start going down as x increases. It sounds like "Oh nooooooooooooooooooooooooo!" or "Piuuuuuuuuuuuuuu". To see that the frequency goes down, maybe plot $\sin(4\sqrt{x})$ on from $x = 0$ to $x = 100$, then imagine what you hear.

2.3. Here is the graph for Problem 3 and 4:

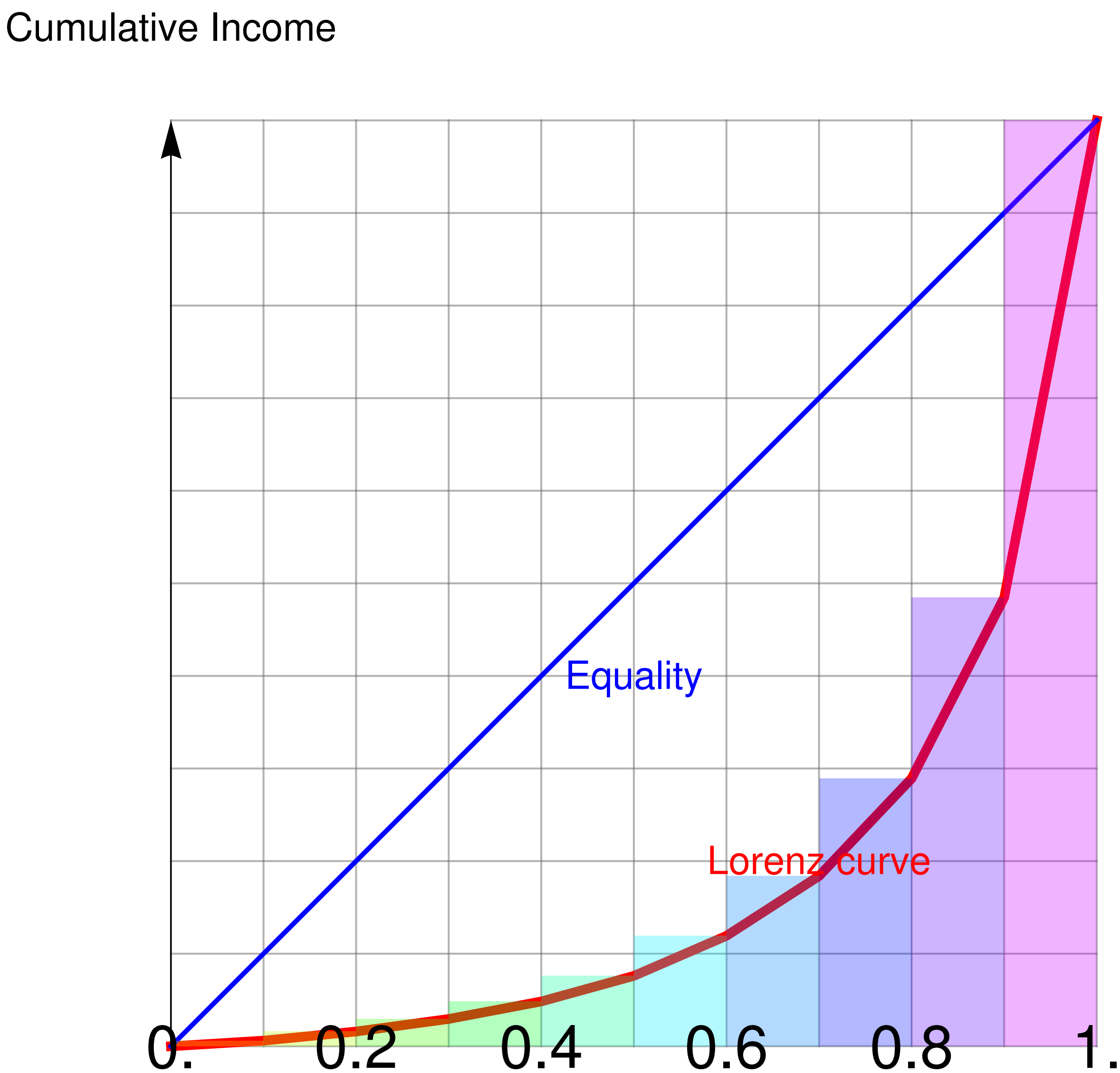

Figure 1. The Lorenz curve.

2.4. The income data points we worked with are:

0	161	260	358	500	730	1142	1707	2784	5161	13613

2.5. This means that the data points for the red curve are (if each square is considered a unit square): (these are the numbers you can work with and which agree with the graph shown above).

0	0.061	0.159	0.295	0.484	0.761	1.193	1.839	2.894	4.847	10.0

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