INTRODUCTION TO CALCULUS

MATH 1A

Unit 1: What is Calculus?

Lecture

1.1. In this welcome lecture we start with a bit of an overview, what calculus is about and **how it can model the world**. Calculus looks at **changes** from the past, to find **models** and **laws**, which then allow to predict the **future**. The analysis of the past uses **differences**, leading to the concept of **derivative**. Solving the model requires **summation**, leading to the concept of **integral**. **Functions** allow to **model** the situation. Their **graphs** bring in some **geometry** as the derivatives lead to **slope** and **concavity** and the integral leads to **area** or **volume**. The following figure shows example of a function, describing a ball bouncing on floor. Can you describe its concavity features?

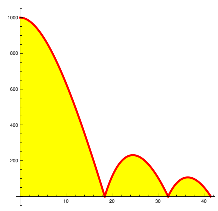

Figure 1. The bouncing of a stone is modeled by a function $f(t)$ which tells how high the ball is at time t. We look at $f(t+h)-f(t)$ to see how f **changes** from t to $t+h$. If we have a law for these changes, we can look into the future and predict where the ball will end up. The **domain** of the function are the t for which the function is defined or considered. In this case, it is $t \in [0,\infty)$ as we do not consider negative time.

1.2. How do we analyze a function? Taking differences can help. Functions $f(x)$ are often given just at a set of points. This produces **data points** like $f(1) = 3, f(2) = 9, f(3) = 19, f(4) = 33, f(5) = 51, f(6) = 73, f(7) = 99$? Can you predict the next term? We will discuss this in class.

1.3. In order to gain visual insight, it is helpful to **draw the data**. We call the picture a **graph**. There are many ways on how one can do that and we will collect some ideas in class. The next figure shows one way to **visualize the data**. It is a **bar chart**.

1.4. The **Fibonnacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ defines a function. We have for example $f(6) = 8$. Can you see the pattern? Can you predict the future and see the next term? Again, if you should not know the rule, look at the rate of change and see whether you can see a rule.

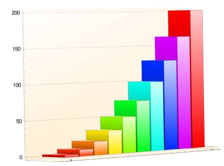

FIGURE 2. When plotting the sequence of numbers in the coordinate plane, the function is visualized as a graph.

HOMEWORK

This problem set is due in grade scope on Wednesday, 1/24/2024 at 9 AM.

Problem 1.1: a) Give a formula of a function $f(x)$ that is decreasing and concave down on the domain $(0, \infty)$.
b) Modify your function in a) so that its graph passes through $(1, 1)$ and $(2, 0)$.

Solution:
a) There are many examples. One example is e^{-x}. An other example is $-x^2$.
b) is a bit tricky if you don't know how to start. One possibility to solve this is to start with a linear function passing through the points, then take a very flat parabola which has roots at 1 and 2. $y = -x + 2$ goes through the points. For example: $f(x) = -x + 2 - (x-1)(x-2)/100$. Here is an other solution. Start with $-x^2$ to have concave down. Now add 4 to have $f(2) = 0$. Now we have $4 - x^2$. In order to get $f(1) = 1$, just divide by 3. A solution is $(4 - x^2)/3$.

Problem 1.2: Give a formula for an example of a function with
a) domain $(-\infty, \infty)$ such that $2 \leq f(x) \leq 3$.
b) domain $(2, \infty)$ which is decreasing and satisfies $f(3) = 1$. (Make sure the function is not defined on $(-\infty, 2]$).
c) domain $(-\infty, \infty)$ which is always decreasing and is always larger than 1.
d) domain $(-\infty, \infty)$ for which $f(2x) = 4f(x)$.

Solution:
Good things to look for: adding a constant moves the graph up or down. Changing $f(x)$ to $f(-x)$ flips around the y-axis. multiplying by a constant squeezes or stretches the graph vertically.
a) $f(x) = 2.5 + \sin(x).2$. Simpler: $f(x) = 2.5$.
b) $f(x) = 1 - \ln(x - 2)$. c) $f(x) = e^{-x} + 1$.
d) $f(x) = x^2$. Just check: $f(2x) = (2x)^2 = 4x^2$.

Problem 1.3: If the Capitol movie theater in Arlington sells a ticket for $x = 10$ dollars, it sells $f(x) = 1000$ tickets. If the ticket prize is increased by 1 to $x = 11$ dollars, the number of tickets decreases by 5. Model the situation with a linear function $f(x) = ax + b$ which predicts the number of tickets sold for prize x.

Solution:
The function has the form $f(x) = ax + b$. We know $f(10) = 1000, f(x + 1) = f(x) - 5$. We see that the slope of the linear function must be -5. So, $f(x) = -5x + b$, where b is some constant. To fix the constant look at $x = 10$. The function $f(x) = -5x + 1050$ works.

Problem 1.4: Galileo experimented with a ball on the inclined plane and measured the distances traversed in equal time. He saw the numbers $1, 3, 5, 7, 9, 11, \ldots$ and called this the **odd number rule**. Find a function $f(x)$ with the property that $g(x) = f(x + 1) - f(x)$ gives the odd numbers $2x + 1$. Describe its concavity nature.

Solution:
In order to get the function, start with 0 and add things up. Function should be $f(0) = 0, f(1) = 1, f(2) = 1 + 3 = 4, f(3) = 1 + 3 + 5 = 9, f(4) = 1 + 3 + 5 + 9 = 16$ etc. We then guess $f(x) = x^2$. It indded has the property $f(x + 1) - f(x) = (x + 1)^2 - x^2 = 2x + 1$. This function x^2 is concave up. The ball is accelerating with a constant rate.

Problem 1.5: Predict the future and find the next term in the sequence

$$2, 10, 30, 68, 130, 222, 350, 520, 738, 1010, 1342, \ldots.$$

Solution:
The next term is 1740.

$$\begin{pmatrix} 2 & 10 & 30 & 68 & 130 & 222 & 350 & 520 & 738 & 1010 & 1342 & 1740 \\ 8 & 20 & 38 & 62 & 92 & 128 & 170 & 218 & 272 & 332 & 398 & 470 \\ 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 & 60 & 66 & 72 & 78 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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