Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

5/8/2024: Final Exam

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

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14		10
Total:		140

Problem 1) TF questions (10 points) No justifications are needed.

1) T F — The fundamental theorem of calculus implies that $\int_{-2}^{2} g'(x)\, dx = g(2) - g(-2)$ for all differentiable functions g.

Solution:
Yes, even if the function is called g.

2) T F — The function $\ln(e^x)$ is defined and continuous for all real numbers x.

Solution:
It simplifies to $\ln(e^x) = x$.

3) T F — $(e^x)^y = (e^y)^x$ for all real numbers x, y.

Solution:
Yes, both are e^{xy}.

4) T F — $\frac{d}{dx} 3^x = x 3^{x-1}$.

Solution:
We would have $3^x ln(3)$

5) T F — The tangent function $f(x) = \tan(x)$ is continuous everywhere.

Solution:
It is not continuous at $x = \pi/2$ for example.

6) T F $\frac{d}{dx}\ln(e^x) = 1$.

Solution:
First simplify $\ln(e^x) = x$. One can also use the chain rule to see it.

7) T F $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

Solution:
Yes, it is the 45-45-90 triangle.

8) T F The function $\cos(x) - 2x$ has a root in the interval $(-7, 7)$.

Solution:
Use the intermediate value theorem.

9) T F The anti-derivative of $1/(1 - x^2)$ is equal to $\arctan(x)$.

Solution:
We woulds need integration by parts to solve this.

10) T F The limit of $\sin^{107}(x)/x^{107}$ for $x \to 0$ exists and is equal to 107.

Solution:
It is equal to 1.

Problem 2) Algebra (10 points)

a) (5 points) Which of the following expressions are integers? If it is an integer, fill in the value.

Expression	Is an integer: give value	Not an integer, no evaluation needed
$\ln(e^{\ln(e^{20})})$		
$\ln(e^2 + e)$		
$\ln(e^7) + \ln(e^9)$		
$\ln(e^e)$		
$e^{\ln(5e)-1})$		

b) (5 points) Which of the following expressions are true for all positive a, b:

Expression	Always true	In general false
$\ln(a + b) = \ln(a) + \ln(b)$		
$e^{a+b} = e^a e^b$		
$\ln(\frac{a}{b}) = \ln(a) - \ln(b)$		
$e^{ab} = e^a e^b$		
$(e^b)^c = e^{bc}$		

Solution:

Expression	Is an integer: give value	Not an integer, no evaluation needed
$\ln(e^{\ln(e^{20})})$	20	
$\ln(e^2 + e)$		x
$\ln(e^7) + \ln(e^9)$	16	
$\ln(e^e)$		x
$e^{\ln(5e)-1})$	5	

Expression	Always true	In general false
$\ln(a + b) = \ln(a) + \ln(b)$		x
$e^{a+b} = e^a e^b$	x	
$\ln(\frac{a}{b}) = \ln(a) - \ln(b)$	x	
$e^{ab} = e^a e^b$		x
$(e^b)^c = e^{bc}$	x	

Problem 3) Functions (10 points)

a) (5 points) In order to find the relation between the following functions, cross each box which applies.

function f	function g	$f = g'$	$g = f'$	none
$\ln(x)$	$\frac{1}{x}$			
$\frac{-1}{x^2}$	$\frac{1}{x}$			
$\tan(x)$	$\frac{1}{1+x^2}$			
$-\frac{1}{\sin^2(x)}$	$\cot(x)$			
$\arctan(x)$	$\frac{1}{\cos^2(x)}$			
$\text{arccot}(x)$	$-\frac{1}{1+x^2}$			

Solution:

function f	function g	$f = g'$	$g = f'$	none
$\ln(x)$	$\frac{1}{x}$		x	
$\frac{-1}{x^2}$	$\frac{1}{x}$	x		
$\tan(x)$	$\frac{1}{1+x^2}$			x
$-\frac{1}{\sin^2(x)}$	$\cot(x)$	x		
$\arctan(x)$	$\frac{1}{\cos^2(x)}$			x
$\text{arccot}(x)$	$-\frac{1}{1+x^2}$		x	

b) (5 points) Which of the following functions has a root at $x = 0$ or $x = \pi$? Check the boxes which apply.

Function f	has a root at x=0	has a root at $x = \pi$
$\sin(x)$		
$\cos(x)$		
$\tan(x)$		
$\ln(1 + x)$		
e^x		

Solution:

Function f	has a root at x=0	has a root at $x = \pi$
$\sin(x)$	x	x
$\cos(x)$		
$\tan(x)$	x	x
$\ln(1 + x)$	x	
e^x		

Problem 4) Continuity (10 points)

We look at five functions for which there are concerns at the point $x = 0$. In each case, you decide whether one can assign a value at $x = 0$ to make the function a continuous function. If yes, state the theorem or method you use and give the value $x = 0$. If the function can not be made continuous, just check the box in the last column.

Function	continuous by (theorem or method)	value at x=0	not continuous
$\sin(\frac{1}{x})$			
$\sin(x)\sin(\frac{1}{x})$			
$\frac{\sin(x)}{x}\sin(x)$			
$\frac{\sin(x)}{x}$			
$\frac{\sin(x)}{x}\sin(\frac{1}{x})$			

Solution:

Function	continuous by (theorem or method)	value at x=0	not continuous
$\sin(\frac{1}{x})$			x
$\sin(x)\sin(\frac{1}{x})$	x Squeeze	0	
$\frac{\sin(x)}{x}\sin(x)$	x Fundamental Thm Trig	0	
$\frac{\sin(x)}{x}$	x Fundamental Thm trig	1	
$\frac{\sin(x)}{x}\sin(\frac{1}{x})$			x

Problem 5) Related Rates (10 points)

a) (5 points) We deal here with two unknown functions $x = x(t), y = y(t)$ related by

$$x + x^2 + 6x^3 + y + y^2 + 3y^3 = 13 \ .$$

You also know $x(0) = 1, y(0) = 1$ and $x'(0) = 5$. What is $y'(0)$?

b) (5 points) Now, we deal with an unknown function $y(x)$. You know again that

$$x + x^2 + 6x^3 + y + y^2 + 3y^3 = 13$$

and that at $x = 1$ we have $y(1) = 1$. What is $y'(1)$?

Solution:
a) This is a related rates problem $-105/12 = -35/4$.
b) This is an implicit differentiation problem $-21/12 = -7/4$.

Problem 6) Integrals (10 points)

Solve the following definite or indefinite integrals:

a) (2 points) $\int_0^\pi \cos(x) - \sin(x)\ dx$

b) (2 points) $\int xe^{3x}\ dx$

c) (2 points) $\int_0^1 \frac{1}{(x+2)^2}\ dx$

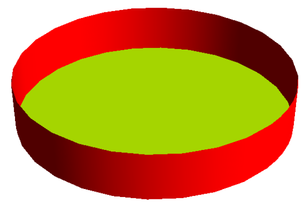
d) (2 points) $\int x\ln(x)\ dx$

e) (2 points) $\int_1^e \ln(x)\ dx$

Solution:
a) -2.
b) $xe^{3x}/3 - e^{3x}/9 + C$
c) $-(x+2)^{-1}|_0^1 = -1/3 + 1/2 = 1/6$
d) $x^2/2\ln(x) - x^2/4 + C$
e) $x\ln(x) - x|_1^e = 1$.

Problem 7) Extrema (10 points)

A **candle holder** of height y and radius x has surface area is $2\pi xy + \pi x^2 = \pi$ so that $y = \frac{1}{2x} - \frac{x}{2}$. We are interested in its volume $f = x^2 y\pi = x^2(\frac{1}{2x} - \frac{x}{2})\pi$. It has been simplified in a).



a) (6 points) Find the point x for which the volume

$$f(x) = \frac{\pi x}{2} - \frac{\pi x^3}{2}$$

is a local maximum. Use the second derivative test to verify.

b) (2 points) On which interval $[a, b]$ does the problem make sense, given that height y and radius x are non-negative?

c) (2 points) Compute the function values $f(a), f(b)$ at the boundary of the interval in part b) to verify that the local maximum you obtained in part a) is also the global maximum.

Solution:
a) The derivative of f is zero for $\boxed{x = \pm 1/\sqrt{3}}$. The second derivative $-3x\pi$ is negative there so that this is a local maximum by the second derivative test. At $-\sqrt{3}$ it would be a global min.
b) The function makes sense if $x \geq 0$ and $y \geq 0$. This means $x \in [0, 1]$.
c) The f values at 0 and 1 are both 0. The only critical point in the interval $[0, 1]$ is $1/\sqrt{3}$. The function has a global max there.

Problem 8) Substitution (10 points)

a) (3 points)

$$\int \frac{\cos(\ln(x))}{x} \, dx \; .$$

b) (4 points)

$$\int e^{\tan(x)} \frac{1}{\cos^2(x)} dx \; .$$

c) (3 points) The infamous sigmoid function

$$\int \frac{e^x}{1+e^x} \, dx \; .$$

Solution:
a) $\sin(\ln(x)) + C$.
b) $e^{\tan(x)} + C$
c) $\ln(1 + e^x) + C$.

Problem 9) Integration by parts (10 points)

a) (5 points) Compute

$$\int x^5 e^{x-3} \, dx \, .$$

b) (5 points) Evaluate the following integral:

$$\int \ln(x) x^4 \, dx \, .$$

Solution:
a) Use the Tic-Tac-Toe method.

x^5	$\exp(x)$	
$5x^4$	$\exp(x-3)$	$\oplus$
$20x^3$	$\exp(x-3)$	$\ominus$
$60x^2$	$\exp(x-3)$	$\oplus$
$120x$	$\exp(x-3)$	$\ominus$
120	$\exp(x-3)$	$\oplus$
0	$\exp(x-3)$	$\ominus$

The answer is $e^{x-3}(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C$.
b) Use integration by parts, and differentiate $\ln(x)$ and get $x^5 \ln(x)/5 - x^5/25 + C$.

Problem 10) Partial fractions (10 points)

a) (5 points) Find the definite integral

$$\int_5^7 \frac{1}{(x-4)(x-2)} \; dx \; .$$

b) (5 points) Find the indefinite integral

$$\int \frac{1}{(x-1)(x-3)(x-7)} \; dx \; .$$

Solution:
In both problems we can find the coefficients quickly with the l'Hopital method: a) $\int_5^7 \frac{1}{(x-4)(x-2)} \; dx = \frac{1}{2} \int_5^7 [\frac{1}{x-4} - \frac{1}{x-2}] \; dx = \frac{1}{2}[\ln|x-4| - \ln|x-2|]|_5^7 = \ln(3) - \ln(5)/2$.
b) The factorization

$$\frac{1}{(x-1)(x-3)(x-7)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x-7}$$

can be obtained quickly from l'Hopital: $A = \lim_{x \to 1} \frac{1}{(x-3)(x-7)} = \frac{1}{12}$ and $B = \lim_{x \to 3} \frac{1}{(x-1)(x-7)} = -\frac{1}{8}$ and $C = \lim_{x \to 7} \frac{1}{(x-1)(x-3)} = \frac{1}{24}$ so that the result is

$$[\ln|x-1|/12 - \ln|x-3|/8 + \ln|x-7|]/24 \; .$$

Problem 11) Applications I (10 points)

a) (2 points) What is the CDF of the PDF f given as $f(x) = 1/x^2$ if $x \geq 1$ and $f(x) = 0$ else? [Give the answer function that holds for $x \geq 1$]

b) (2 points) If x is the Midi number, then $f(x) = 440 \cdot 2^{\frac{x-69}{12}}$ is called the corresponding [give the expression, one word.]

c) (2 points) If $f(x)$ is a PDF, then $M_n = \int_{-\infty}^{\infty} x^n f(x)\, dx$ is called a [give the expresion, one word.]

d) (2 points) For the family of functions $f_c(x) = c\cos(x)$, there is a catastrophe at $c =$ [give a number, one number.]

e) (2 points) Any continuous function can be approximated by a linear combination of network function $\sigma(ax + b)$. This fact is used in modern transformer networks and called the [give the missing word]

approximation theorem.

Solution:
a) $F(x) = \int_1^x 1/t^2 \, dt = 1 - 1/x$.
b) The frequency.
c) The n'th moment.
d) $c = 0$.
e) Universal

Problem 12) Applications II (10 points)

If $F(x) = x^3 \ln(x)$ is the total cost of a good and $f(x) = F'(x) = 3x^2 \ln(x) + x^2$ and $g(x) = x^2 \ln(x)$.

a) (2 points) What is the technical name for $f(x)$ used in economics?

b) (2 points) What is the technical name for $g(x)$ used in economics?

c) (2 points) Find the non-zero break-even point $f = g$.

d) (2 points) For which positive x is g maximal?

e) (2 points) Sweet surprise: what theorem assures that the results in c) and d) the same?

Solution:
a) Marginal cost.
b) Average cost.
c) $f = g$ for $x^2 + 2x^2 \ln(x) = 0$ meaning $x = 1/\sqrt{e}$.
d) $g' = 2x \ln(x) + x = 0$ means $x = 1/\sqrt{e}$.
e) Strawberry!

Problem 13) Definitions (10 points)

Please complete the sentences with one or two words. Each question is one point.

$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is called the		of f.
$f'(x) = 0, f''(x) > 0$ implies that x is a		of f.
The sum $\frac{1}{n}[f(0) + f(1/n) + f(2/n) + \cdots + f((n-1)/n) + f(1)]$ is called a		sum.
If $f(0) = -3$ and $f(4) = 8$, then f has a root on the interval $(0, 4)$ by the		theorem.
There is a point $x \in (0, 1)$ where $f'(x) = f(1) - f(0)$ by the		theorem.
When finding the derivative of an inverse function like $\arccos'(x)$ or $\text{arctanh}'(x)$, we have used the		rule.
The anti derivative $\int_{-\infty}^{x} f(t)\, dt$ of a probability density function f is called the		function.
A point x for which $f(x) = 0$ is called a		of f.
A point x for which $f'(x) = 0$ is called a		of f.
At a point x for which $f''(x) > 0$, the function is called		up.

Solution:
Derivative
Local minimum
Riemann sum
Intermediate value
Mean value
Chain rule
Cumulative distribution
Root
Critical point Concave

Problem 14) Theorems (10 points)

a) (4 points) Which of the following statements A)-D) are true and belong to the fundamental theorem of calculus.

A)	$\int_0^x f'(t)\, dt = f(x) - f(0)$
B)	$\int_a^b f(x)\, dx = f(b) - f(a)$
C)	$\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$
D)	$\int_a^b f'(x)\, dx = f(b) - f(a)$

Enter one or more letters from A-D

b) (4 points) Which of the following statements E)-H) are true for all continuous functions and do not require the function to be differentiable: Which ones?

E)	Extremal value theorem
F)	Mean value theorem
G)	Intermediate value theorem
H)	Rolle's theorem

Enter one or more letters from E-H here:

c) (2 points) Which of the theorems E)-H) was used to prove the fundamental theorem of calculus? Enter just one of the letters E)-H) here:

Solution:
a) ACD
b) EG
c) F