Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

5/8/2024: Final Exam Practice C

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

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Total:		140

Problem 1) TF questions (10 points) No justifications are needed.

1) [T] [F] The function $x + \sin(\cos(\sin(x)))$ has a root in the interval $(-10, 10)$.

2) [T] [F] The function $1/\log(2 - |x|)$ is defined and continuous for all real numbers x.

3) [T] [F] The Newton iteration method allows to find the roots for any continuous function.

4) [T] [F] The logarithm function $\log(x)$ is monotonically increasing for all $x > 0$.

5) [T] [F] A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

6) [T] [F] If the total cost $F(x)$ of an entity is extremal at x, then we have a break even point $f(x) = g(x)$.

7) [T] [F] The value $\int_{-\infty}^{\infty} xf(x)\, dx$ is called the expectation of the PDF f.

8) [T] [F] $\tan(\pi/3) = \sqrt{3}$.

9) [T] [F] The limit of $\sqrt{|x|}/\sin(\sqrt{|x|})$ for $x \to 0$ exists and is equal to 1.

10) [T] [F] $\sin(\arctan(1)) = \sqrt{3}$.

Problem 2) Algebra (10 points)

Solve the following equations for x.

a) $x^4 + 1 = 2x^2$

b) $\sin(x) = \sqrt{3}/2$

c) $7^x = 1$

d) $\cot(x) = \cos(x)$

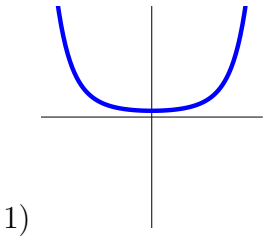
e) $\sqrt{x} + x = x\sqrt{x}$.

Problem 3) Functions (10 points)

a) (5 points) Sketch the graphs of following functions $\sin(x), \cos(x), \tan(x), \exp(x), \ln|x|$ on the interval $[-\pi, \pi]$ as well as you can. Make sure to indicate the roots (if there exists one) and the vertical asymptotes (if there is one) and label the x places for roots and asymptotes. For the function $1/x$ for example, you would draw the hyperboloid and the vertical asymptote at $x = 0$ and indicate that there is no root.

b) (5 points) Match the name of the functions with their graphs (1-4), with their derivatives (A-D) (middle row) and with the second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$1/(1+x^2)$			
$\sin(x^3)$			
$x^3 - x$			
$\frac{e^{1+x^2}}{20}$			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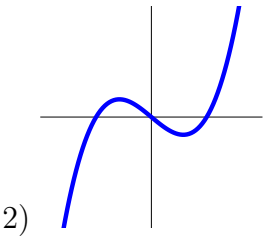
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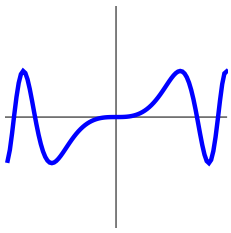
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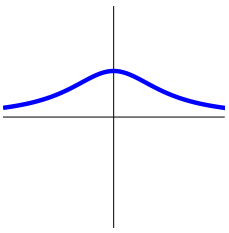
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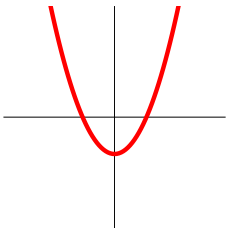
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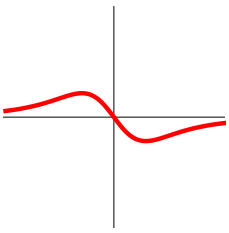
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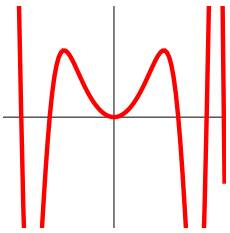
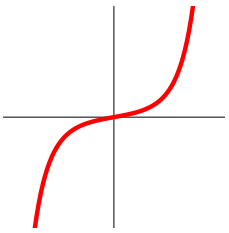

C)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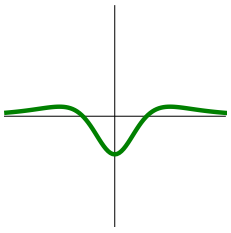
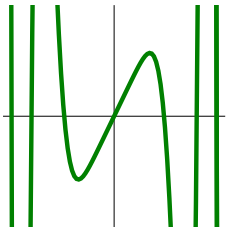

a)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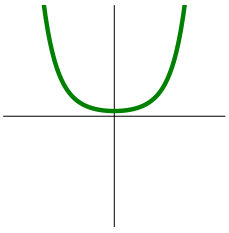
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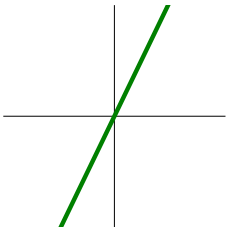

d)

Problem 4) Limits, Continuity (10 points)

a) Find the limits or indicate if the limit should not exist

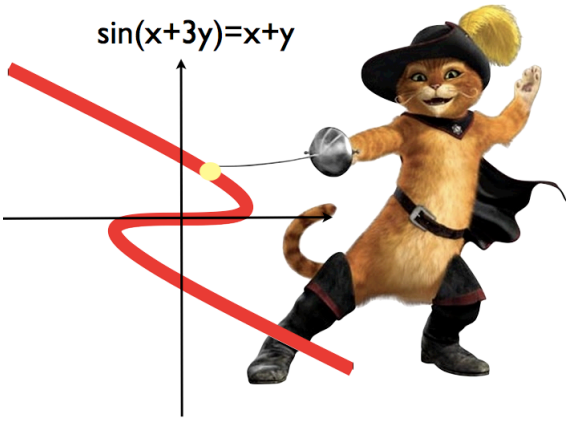
$\lim_{x\to 0} x^2/\sin^2(x)$	
$\lim_{x\to \infty} \sin^2(x)/x^2$	
$\lim_{x\to \infty} \ln(x)/x$	
$\lim_{x\to 0} \ln\lvert x\rvert/x$	
$\lim_{x\to 0} x/\ln\lvert x\rvert$	

b) (5 points) In which cases can we take the limit $x \to 0$? If there is a limit, enter it in the left column, otherwise cross check the right column. If you write on your own paper, please copy the table first.

Function	The limit is (if it exists)	Cross check if not existing
$\frac{\sin(17x)}{\sin(23x)}$		
$-x\log\lvert 3x\rvert$		
$\frac{\sin(x^2)}{\sin^2(x)}$		
$\log\lvert 5x\rvert/\log\lvert 7x\rvert$		
$\arctan(x)/\tan(x)$		
$\frac{\cos(x)+1}{x^2}$		

Problem 5) Related Rates (10 points)

a) Find the derivative of $f(g(x))$, where $f(x) = \sin(\pi x)$ and $g(x) = x^4 + 3x$. b) Let us look at a specific point x. While x is unknown, you know $g(x) = 4$ and $g'(x) = 7$. What is $\frac{d}{dx} f(g(x))$ at this point?



Problem 6) Integrals (10 points)

Which integral method is used?

a) (5 points) Find the anti-derivative of

$$\int e^{e^x} e^x \, dx \, .$$

b) (5 points) And what is the anti-derivative of

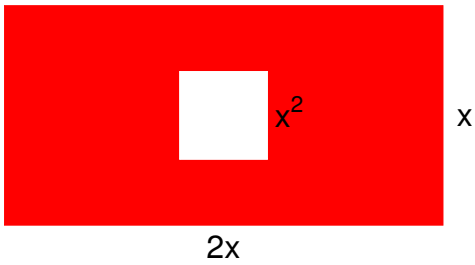
$$\int (\log(x))^2 x \, dx \, .$$

Problem 7) Extrema (10 points)

We want to find the maximal area of a rectangle of length $2x$ and height x in which a square hole of length x^2 has been taken out. The area function is

$$f(x) = 2x^2 - x^4 \ .$$

Use the second derivative test to locate the maximum.



Problem 8) Substitution (10 points)

a) (3 points) Solve the integral $\int \log(x^3)x^2 \, dx$.

b) (4 points) Solve the integral $\int x \cos(x^2) \exp(\sin(x^2)) \, dx$.

c) (3 points) Find the integral $\int \sin(\exp(x)) \exp(x) \, dx$.

Problem 9) Integration by parts (10 points)

a) (5 points) Find

$$\int (x+5)^3 \sin(x-4)\, dx\ .$$

b) (5 point) Find the indefinite integral

$$\int e^x \cos(2x)\, dx\ .$$

Don't get dizzy when riding this one.

Problem 10) Fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-2)(x-3)(x-4)}\, dx\ .$$

(Evaluate the absolute values $\log|\cdot|$ in your answer. The improper integrals exist as a Cauchy principal value).

b) (5 points) Find the indefinite integral

$$\int \frac{1}{x(x-1)(x+1)(x-2)}\, dx\ .$$

Problem 11) Applications I (10 points)

a) (4 points) Complete the following table of probability distributions and cumulative distribution functions.

PDF	PDF supported on	CDF on that interval
e^{-x}	$[0, \infty)$	
	$(-\pi/2, \pi/2)$	$\frac{\arcsin(x)}{\pi} + \frac{1}{2}$
	$(-\infty, \infty)$	$\frac{\arctan(x)}{\pi} + \frac{1}{2}$

b) (3 points) If f is the marginal cost and F the total cost and g the average cost. What is the definition of the **break even point** in this context?

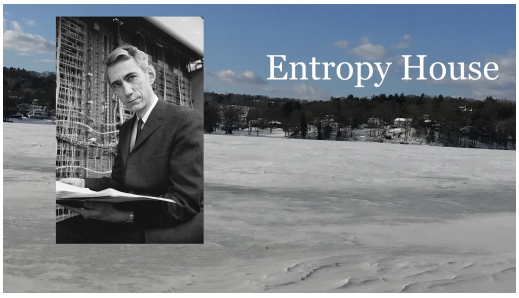
c) (3 points) What theorem is responsible for the fact that there is a point on earth such that the temperature on P and its anti-pod point Q are exactly the same?

Problem 12) Applications II (10 points)

If $f(x)$ is a PDF, then

$$S = -\int_{\infty}^{\infty} f(x) \log(f(x))\, dx$$

is called the **entropy** of f.
What is the entropy of the exponential distribution, given by the function which is 0 for negative x and e^{-x} for $x \geq 0$?



The Entropy house in Winchester, MA on Mystic Lake, where Claude Shannon, the father of information theory lived. Photo: Oliver Knill, 2018.

Problem 13) Definitions (10 points)

a) (2 points) Let $f_c(x)$ denote the family of functions $f_c(x) = cx^4 - c$. Then $c = 0$ is called a

b) (2 points) If we listen to $f(x) = |\sin(x)|\sin(1000x)$ then $|\sin(x)|$ is called the

c) (2 points) If $F(x)$ is the total cost of x goods, then $F'(x)$ is called the

d) (2 points) The function $(e^x + e^{-x})/2$ is also called the .... .

e) (2 points) State the quotient rule:

Problem 14) Theorems (10 points)

Name dropping: Match results with names

Result	Enter A-H
Fundamental theorem of trigonometry	
Universal approximation theorem	
Newton step	
Fundamental theorem of calculus	
Mean value theorem	
Rolle's theorem	
Intermediate value theorem	
Fermat theorem	

A)	$\int_0^1 f'(x)\,dx = f(1) - f(0)$
B)	$\lim_{x\to 0} \sin(x)/x = 1$
C)	$f(0) = -1, f(1) = 1$ implies $f(x) = 0$ for some $x \in (0, 1)$.
D)	f is continuous on $[0, 1]$ then f has a global max and min on $[0, 1]$.
E)	$T(x) = x - f(x)/f'(x)$.
F)	If $f(0) = f(1) = 0$ then $f'(x) = 0$ for some $x \in (0, 1)$.
G)	There exists x in $(0, 1)$ such that $f'(x) = f(1) - f(0)$.
H)	Sums of neural network functions approximate any function.