Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

5/8/2024: Final Exam Practice C

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

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2		10
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8		10
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12		10
13		10
14		10
Total:		140

Problem 1) TF questions (10 points) No justifications are needed.

1) T F

The function $x + \sin(\cos(\sin(x)))$ has a root in the interval $(-10, 10)$.

Solution:
Use the intermediate value theorem.

2) T F

The function $1/\log(2 - |x|)$ is defined and continuous for all real numbers x.

Solution:
It is not defined at $x = 2$ and $x = -2$.

3) T F

The Newton iteration method allows to find the roots for any continuous function.

Solution:
The function has to be differentiable in order to apply Newton.

4) T F

The logarithm function $\log(x)$ is monotonically increasing for all $x > 0$.

Solution:
Yes, the derivative is $1/x$ which is positive for positive x.

5) T F

A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

Solution:
True, we have seen that.

6) T **F** If the total cost $F(x)$ of an entity is extremal at x, then we have a break even point $f(x) = g(x)$.

Solution:
This is not the strawberry theorem.

7) **T** F The value $\int_{-\infty}^{\infty} x f(x)\, dx$ is called the expectation of the PDF f.

Solution:
Yes this is true

8) **T** F $\tan(\pi/3) = \sqrt{3}$.

Solution:
Yes, it is equal to $\sin(\pi/6)$.

9) **T** F The limit of $\sqrt{|x|}/\sin(\sqrt{|x|})$ for $x \to 0$ exists and is equal to 1.

Solution:
It is equal to 1 because of the fundamental theorem of trig. Maybe just put $u = \sqrt{|x|}$.

10) T **F** $\sin(\arctan(1)) = \sqrt{3}$.

Solution:
We have $\arctan(1) = \pi/4$ and so $\sin(\arctan(1)) = \sqrt{3}$.

Problem 2) Algebra (10 points)

Solve the following equations for x.

a) $x^4 + 1 = 2x^2$

b) $\sin(x) = \sqrt{3}/2$

c) $7^x = 1$

d) $\cot(x) = \cos(x)$

e) $\sqrt{x} + x = x\sqrt{x}$.

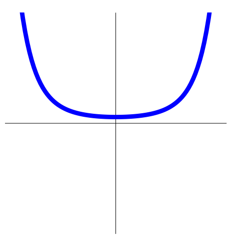
Solution:
a) $x = 1$ or $x = -1$
b) $\pi/3 = 60°$.
c) $x = 0$.
d) $\sin(x) = 1$ which means $x = \pi/2 + k2\pi$ with integer k.
e) $(\sqrt{5} + 3)/2$.

Problem 3) Functions (10 points)

a) (5 points) Sketch the graphs of following functions $\sin(x), \cos(x), \tan(x), \exp(x), \ln|x|$ on the interval $[-\pi, \pi]$ as well as you can. Make sure to indicate the roots (if there exists one) and the vertical asymptotes (if there is one) and label the x places for roots and asymptotes. For the function $1/x$ for example, you would draw the hyperboloid and the vertical asymptote at $x = 0$ and indicate that there is no root.

b) (5 points) Match the name of the functions with their graphs (1-4), with their derivatives (A-D) (middle row) and with the second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$1/(1+x^2)$			
$\sin(x^3)$			
$x^3 - x$			
$\frac{e^{1+x^2}}{20}$			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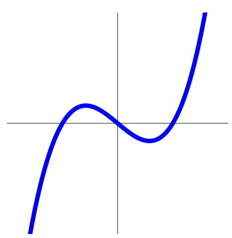
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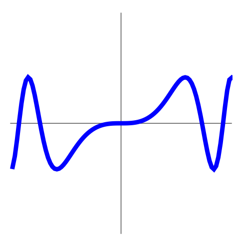
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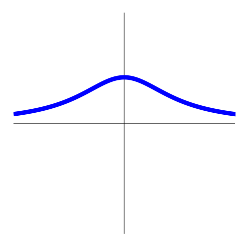
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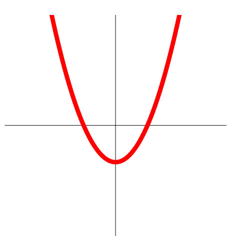
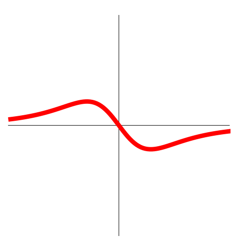

A)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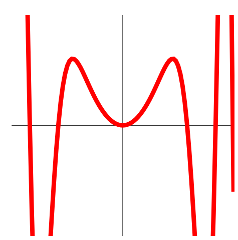
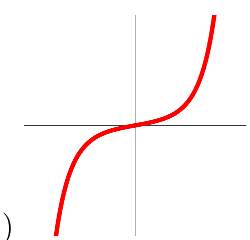

C)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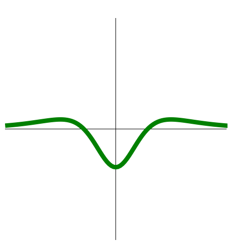
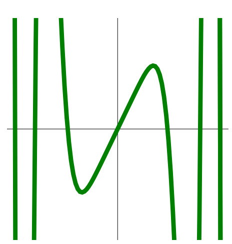

a)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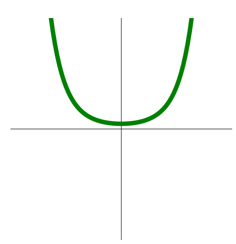
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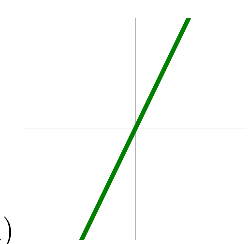

d)

Solution:
a) The sin function has roots at 0 and pi. The cos function has roots at $-\pi/2, \pi/2$ The tan function has roots at 0 and π and asymptotes at $-\pi/2, \pi/2$.
The exp function has no roots and no asymptotes. The log function has roots at -1,1 and an asymptote at x-0.
4 B a
3 C b
2 A d
1 D c

Problem 4) Limits, Continuity (10 points)

a) Find the limits or indicate if the limit should not exist

$\lim_{x\to 0} x^2/\sin^2(x)$	
$\lim_{x\to\infty} \sin^2(x)/x^2$	
$\lim_{x\to\infty} \ln(x)/x$	
$\lim_{x\to 0} \ln\lvert x\rvert/x$	
$\lim_{x\to 0} x/\ln\lvert x\rvert$	

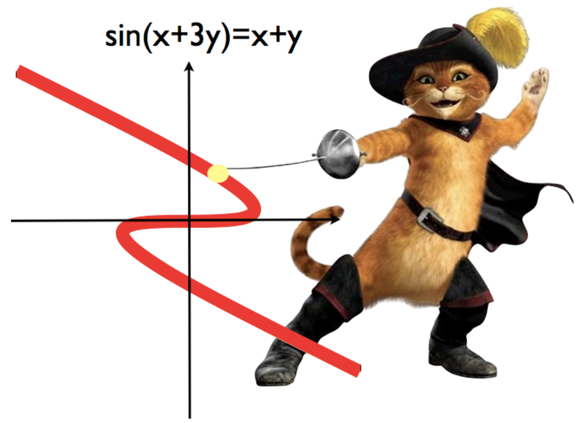
b) (5 points) In which cases can we take the limit $x \to 0$? If there is a limit, enter it in the left column, otherwise cross check the right column. If you write on your own paper, please copy the table first.

Function	The limit is (if it exists)	Cross check if not existing
$\frac{\sin(17x)}{\sin(23x)}$		
$-x\log\lvert 3x\rvert$		
$\frac{\sin(x^2)}{\sin^2(x)}$		
$\log\lvert 5x\rvert/\log\lvert 7x\rvert$		
$\arctan(x)/\tan(x)$		
$\frac{\cos(x)+1}{x^2}$		

Solution:
a) In all cases, use Hospital. We have 1,0,0, DNE, 0.
b) 17/23,0,1,1,1,DNE

Problem 5) Related Rates (10 points)

a) Find the derivative of $f(g(x))$, where $f(x) = \sin(\pi x)$ and $g(x) = x^4 + 3x$. b) Let us look at a specific point x. While x is unknown, you know $g(x) = 4$ and $g'(x) = 7$. What is $\frac{d}{dx} f(g(x))$ at this point?



Solution:
a) The derivative is $\pi\cos(\pi(x^4 + 3x))(4x^3)$.
b) It is $\pi\cos(4\pi)7 = 7\pi$.

Problem 6) Integrals (10 points)

Which integral method is used?

a) (5 points) Find the anti-derivative of

$$\int e^{e^x} e^x \; dx \; .$$

b) (5 points) And what is the anti-derivative of

$$\int (\log(x))^2 x \; dx \; .$$

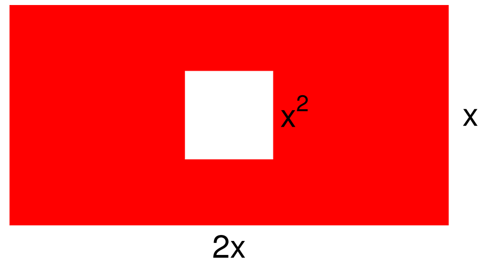
Solution:
a) Make a substitution $u = e^x, du = e^x dx$ to get $\int e^u \; du = e^u + c = e^{e^x} + c$.
b) Use Integration by parts $u = \log(x), dv = xdx$ so that we have $\log(x)^2 x^2/2 - \int \log(x)x/; dx$. Now use integration by parts again for computing $\int \log(x)xdx$ which is $x^2 \log(x)/2 - x^2/4$. The final answer is $x^2 \log(x)^2/2 - x^2 \log(x)/2 + x^2/4$.

Problem 7) Extrema (10 points)

We want to find the maximal area of a rectangle of length $2x$ and height x in which a square hole of length x^2 has been taken out. The area function is

$$f(x) = 2x^2 - x^4 \ .$$

Use the second derivative test to locate the maximum.



Solution:
Take the derivative $f'(x) = 4x - 4x^3$ and set it to zero. The roots are $0, 1, -1$. We have $f''(1) = -8, f''(-1) = -8, f''(0) = 4$ so that $1, -1$ are maxima and 0 is a minimum. As we deal with a geometric situation $x = 1$ is the solution.

Problem 8) Substitution (10 points)

a) (3 points) Solve the integral $\int \log(x^3)x^2 \, dx$.

b) (4 points) Solve the integral $\int x\cos(x^2)\exp(\sin(x^2)) \, dx$.

c) (3 points) Find the integral $\int \sin(\exp(x))\exp(x) \, dx$.

Solution:
These are all standard substitution problems:
a) Substitute $u = x^3$ to get $(x^3\log(x^3) - x^3)/3 + C$
b) Substitute $u = \sin(x^2), du = \cos(x^2)2x$ to get $\exp(\sin(x^2))/2 + C$.
c) Substitute $u = \exp(x)$ to get $-\cos(\exp(x)) + C$.

Problem 9) Integration by parts (10 points)

a) (5 points) Find

$$\int (x+5)^3 \sin(x-4)\, dx \; .$$

b) (5 point) Find the indefinite integral

$$\int e^x \cos(2x)\, dx \; .$$

Don't get dizzy when riding this one.

Solution:
a) Use the Tic-Tac-Toe integration method:

$(x+5)^3$	$\sin(x-4)$	
$3(x+5)^2$	$-\cos(x-4)$	$\oplus$
$6(x+5)^1$	$-\sin(x-4)$	$\ominus$
6	$\cos(x-4)$	$\oplus$
0	$\sin(x-4)$	$\ominus$

We can read off the answer $-(x-5)^3\cos(x-4) + 3(x+5)^2\sin(x-4) + 6(x+5)\cos(x-4) - 6\sin(x-4) + C$.
b) We use the merry go round by using integration by parts twice calling the integral I. We have

$$I = \cos(2x)e^x + \int 2\sin(2x)e^x\, dx = \cos(2x) + 2\sin(2x) - 4I \; .$$

Solving for I gives $\boxed{(\cos(x) + 2\sin(2x))e^x/5 + C}$.

Problem 10) Fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-2)(x-3)(x-4)} \, dx \; .$$

(Evaluate the absolute values $\log|\cdot|$ in your answer. The improper integrals exist as a Cauchy principal value).

b) (5 points) Find the indefinite integral

$$\int \frac{1}{x(x-1)(x+1)(x-2)} \, dx \; .$$

Solution:
a) We use the hopital method to find the constants $A = 1/2, B = -1, C = 1/2$ in

$$\frac{1}{(x-2)(x-3)(x-4)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} + \frac{C}{(x-4)} \; .$$

The answer is $\log(x-2)/2 - \log(x-3) + \log(x-4)/2$. The definite integral is $\boxed{\text{zero}}$.
b) Again use Hopital to get $A = 1/2, B = -1/2, C = -1/6, D = 1/6$ in

$$\frac{1}{x(x-1)(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x-2} \; .$$

The answer is

$$\log(x)/2 - \log(x-1)/2 - \log(x+1)/6 + \log(x-2)/6 + C \; .$$

Problem 11) Applications I (10 points)

a) (4 points) Complete the following table of probability distributions and cumulative distribution functions.

PDF	PDF supported on	CDF on that interval
e^{-x}	$[0, \infty)$	
	$(-\pi/2, \pi/2)$	$\frac{\arcsin(x)}{\pi} + \frac{1}{2}$
	$(-\infty, \infty)$	$\frac{\arctan(x)}{\pi} + \frac{1}{2}$

b) (3 points) If f is the marginal cost and F the total cost and g the average cost. What is the definition of the **break even point** in this context?

c) (3 points) What theorem is responsible for the fact that there is a point on earth such that the temperature on P and its anti-pod point Q are exactly the same?

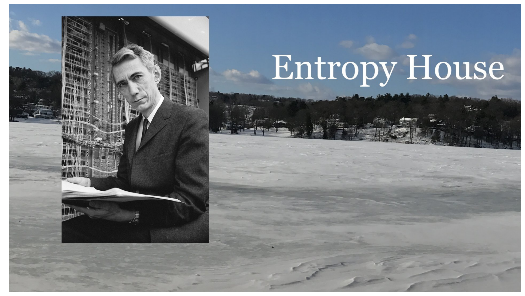
Solution:
a) $1 - e^{-x}$, $(1 - x^2)^{-1/2}/\pi$, $(1 + x^2)^{-1}/\pi$ c) f=g
d) Intermediate value theorem

Problem 12) Applications II (10 points)

If $f(x)$ is a PDF, then

$$S = -\int_{\infty}^{\infty} f(x) \log(f(x))\, dx$$

is called the **entropy** of f.
What is the entropy of the exponential distribution, given by the function which is 0 for negative x and e^{-x} for $x \geq 0$?



The Entropy house in Winchester, MA on Mystic Lake, where Claude Shannon, the father of information theory lived. Photo: Oliver Knill, 2018.

Solution:

$$S = -\int_0^{\infty} f(x) \log(f(x))\, dx = -\int_0^{\infty} e^{-x}(-x)\, dx$$

This is $\int_0^\infty e^{-x} x\, dx$ which is solved by integration by parts by differentiating $x = u$ and integrating $e^{-x} dx = dv$. The result of this improper integral is 1. We have seen this integral already as the expectation of the exponential distribution.
By the way: This is an important computation. It turns out that from all nice probability distributions on $[0, \infty)$ with mean 1 and variance 1, the exponential distribution is the one which maximizes entropy. On the $(-\infty, \infty)$, among all distributions with mean 0 and variance 1 the normal distribution $f(x) = e^{-x^2/2}/\sqrt{2\pi}$ maximizes entropy.

Problem 13) Definitions (10 points)

a) (2 points) Let $f_c(x)$ denote the family of functions $f_c(x) = cx^4 - c$. Then $c = 0$ is called a

b) (2 points) If we listen to $f(x) = |\sin(x)|\sin(1000x)$ then $|\sin(x)|$ is called the

c) (2 points) If $F(x)$ is the total cost of x goods, then $F'(x)$ is called the

d) (2 points) The function $(e^x + e^{-x})/2$ is also called the .... .

e) (2 points) State the quotient rule:

Solution:
a) Catastrophe.
b) Hull function.
c) Marginal cost
d) cosh
e) $(f/g)' = (f'g - fg')/g^2$

Problem 14) Theorems (10 points)

Name dropping: Match results with names

Result	Enter A-H
Fundamental theorem of trigonometry	
Universal approximation theorem	
Newton step	
Fundamental theorem of calculus	
Mean value theorem	
Rolle's theorem	
Intermediate value theorem	
Fermat theorem	

A)	$\int_0^1 f'(x)\,dx = f(1) - f(0)$
B)	$\lim_{x\to 0} \sin(x)/x = 1$
C)	$f(0) = -1, f(1) = 1$ implies $f(x) = 0$ for some $x \in (0,1)$.
D)	f is continuous on $[0,1]$ then f has a global max and min on $[0,1]$.
E)	$T(x) = x - f(x)/f'(x)$.
F)	If $f(0) = f(1) = 0$ then $f'(x) = 0$ for some $x \in (0,1)$.
G)	There exists x in $(0,1)$ such that $f'(x) = f(1) - f(0)$.
H)	Sums of neural network functions approximate any function.

Solution:
B) H) E) A) G) C) F)