Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

5/8/2024: Final Exam Practice B

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

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2		10
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12		10
13		10
14		10
Total:		140

Problem 1) TF questions (10 points) No justifications are needed.

1) T F The limit $\lim_{x \to 0} x \ln(|x|)$ is 0.

2) T F One of the double angle formulas is $\sin(2x) = \frac{(1-\cos^2(x))}{2}$

3) T F The function $f(x) = x(1-x^5) + \sin(\pi x)$ has a critical point in $(0, 1)$.

4) T F The function $f(x) = 1 + \sin(x^2) - x^3$ has a root in the interval $(-100, 100)$.

5) T F To any continuous function f, there exists a unique $F(x)$ such that $F'(x) = f(x)$.

6) T F If you listen to the sound $\ln(1+x)\sin(10000x)$, then it gets louder and louder as time goes on.

7) T F The function $f(x) = (x^{25} - 1)/(x^5 - 1)$ has the limit 20 for $x \to 1$.

8) T F If the average cost $F(x)/x$ of an entity is extremal at $x = 2$, then we have a break-even point $f(2) = g(2)$.

9) T F The Midi function $f(s)$ gives the midi number $f(s)$ as a function of the frequency s.

10) T F A Newton step for the function f is $T(x) = x + \frac{f(x)}{f'(x)}$.

Problem 2) Algebra (10 points)

Simplify the following expressions until they become a quantity that does not depend on x any more.

a) $\frac{1}{\frac{1}{x}} - x$

b) $(1/x)/(x/x^2)$

c) $1 + \sqrt{x^6}x^{-3}$

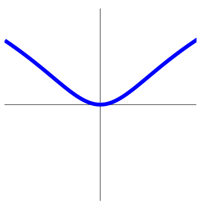
d) $(2^x)^2 - (2^2)^x$

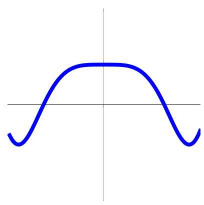
e) $\ln(e^{x+3}e^{-x})$.

Problem 3) Functions (10 points)

a) (5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)$			
$\cos(x^2)$			
$\ln(1+x^2)$			
$\exp(-x^2)$			


1)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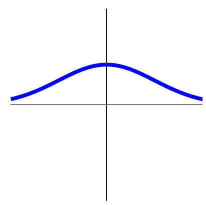
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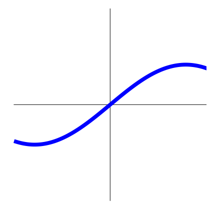
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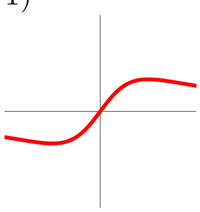
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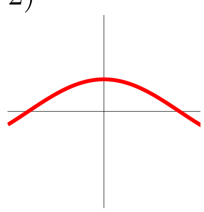
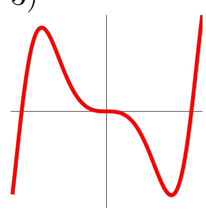
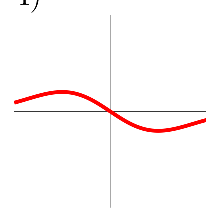
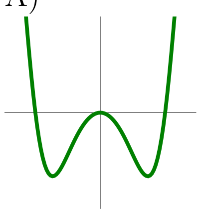
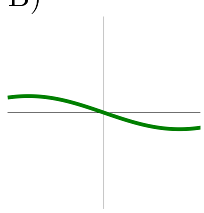

B)


C)


D)


a)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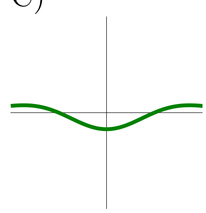
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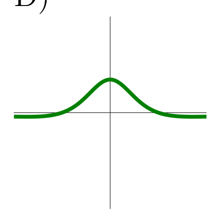

d)

Problem 4) Limits, Continuity (10 points)

Which of the following limits exist in the limit $x \to 1$? If the limit exists, enter the result:

Function	Enter the limit if it exists	Check if it does not exist
$\frac{(1-x^2)}{(1-x)}$		
$\frac{\ln\|1-x\|}{\ln(1+x)}$		
$\frac{(1-x^9)}{(1-x^7)}$		
$\frac{x}{\ln\|x\|}$		
$\frac{\tan(1-x)}{(1-x)}$		
$\ln\|x\|/x$		
$\ln(x)/\ln(2x)$		
$\frac{x^2-1}{\sin(x^2-1)}$		
$\frac{\cot(1-x)}{x}$		
$\frac{\text{sinc}(x)}{1-x}$		

Problem 5) Related Rates (10 points)

a) (7 points) We know that

$$yx^3 + 2xy^3 + xy = 4$$

relates two functions $x = x(t), y = y(t)$ and that $x'(0) = 4$ and $x = x(0) = 1, y = y(0) = 1$. Find $y' = y'(0)$?

b) (3 points) For the same relation but if $y = y(x)$ we want to know $y'(x)$ at $x = 1$, $y = 1$.

Problem 6) Integrals (10 points)

a) (5 points) Determine the value of the definite integral

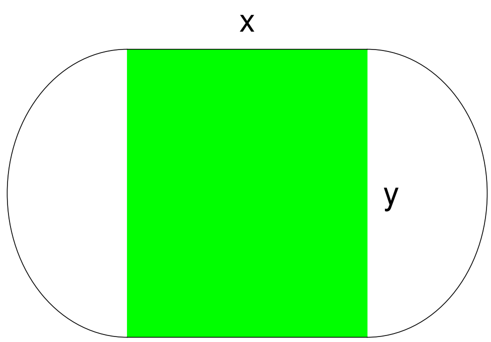
$$\int_0^{\pi/2} e^{\sin(x)} \cos(x) \, dx \; .$$

b) (5 points) Find the anti-derivative

$$\int (1+x)^3 + \frac{1}{1+x^2} + \frac{1}{\sqrt{1-x^2}} \, dx \; .$$

Problem 7) Extrema (10 points)

The Harvard stadium has a track which encloses a rectangular field of dimensions x, y. The circumference of the track is $400 = 2\pi y + 2x$ and is fixed. We want to maximize the area xy for a play field. Which x achieves this?



Problem 8) Substitution (10 points)

a) (5 points) **"One,Two,Three,Four Five, once I caught a fish alive!"**

$$\int \frac{(1+2x+3x^2+4x^3+5x^4)}{(1+x+x^2+x^3+x^4+x^5)} \, dx \; .$$

b) (5 points) A **"Trig Trick-or-Treat"** problem:

$$\int (1-x^2)^{-3/2} + (1-x^2)^{-1/2} + (1-x^2)^{1/2} \, dx \; .$$

Problem 9) Integration by parts (10 points)

a) (5 points) Find

$$\int (1 + x + x^2 + x^3 + x^4)(\sin(x) + e^x)\, dx \ .$$

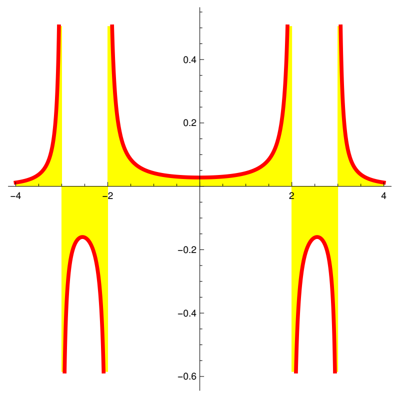
b) (5 points) Find

$$\int \ln(x) \frac{1}{x^2}\, dx \ .$$

Problem 10) Fractions (10 points)

Integrate

$$\int_{-1}^{1} \frac{1}{(x+3)(x+2)(x-2)(x-3)} \, dx \ .$$



The graph of the function is shown to the right.

Lets call it the **friendship graph**.

Problem 11) Applications I. (10 points)

We look at two functions which are defined to be zero outside the interval $[0, \pi]$ and which are given on $0 \leq x \leq \pi$ as

$$f(x) = \frac{\sin(x)}{2}$$

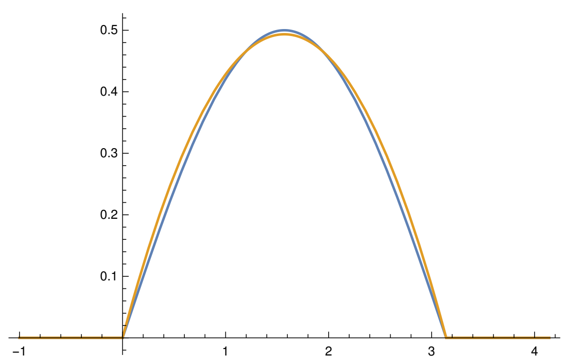
and

$$g(x) = \frac{x(\pi - x)}{5} .$$

Their graphs are very close to each other, as you can see in the picture.

a) (5 points) One of these functions produces a probability distribution. Which one? Justify your answer.

b) (5 points) The other function needs to be multiplied with a constant c to become a probability distribution. What is this constant?



Problem 12) Applications II (10 points)

a) (2 points) What is the expectation $\int x f(x)\, dx$ of the probability distribution function $f(x) = e^{-|x|}/2$ which is a function defined for all real x. [We only need a number.]

b) (2 points) What is the inverse function of the (fictional) **Stockhausen Midi function** $f(x) = 440 \cdot 5^{(x-25)/5}$? This function gives the midi number x as a function of the frequency f.

d) (2 points) The family of functions $f_c(x) = x^4 - cx^2 + x^2$ experiences a **catastrophe**. What is the parameter c for which this happens? [You only need to find that parameter.]

e) (2 points) If $(-2, -7)$ and $(2, 7)$ are two data points. What is the best linear fit $y = ax + b$ which passes through the two points?

Problem 13) Definitions (10 points)

a) (2 points) The logarithm is defined as the

of the exponential function.

b) (2 points) An expression of the form "0/0" or "∞/∞" is called

.

c) (2 points) If we simplify $f(x) = (x^2 - 1)/(x - 1) = x + 1$ we called this process also called

d) (2 points) If an function value is overall the largest on $[0, 1]$ it is called a [fill in the adjective]

maximum.

e) (2 points) A function F satisfying $F' = f$ is called

Problem 14) Theorems (10 points)

a) State both forms of the fundamental theorem of calculus.

b) State the first and second derivative test.

c) State the intermediate value theorem .

d) State the mean value theorem.

e) State the extremal value theorem.