Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

5/8/2024: Final Exam Practice B

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

1		10
2		10
3		10
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10		10
11		10
12		10
13		10
14		10
Total:		140

Problem 1) TF questions (10 points) No justifications are needed.

1) T F

The limit $\lim_{x \to 0} x \ln(|x|)$ is 0.

Solution:
We did that several times. It follows from l'Hospital by writing first $\ln|x|/(1/x)$.

2) T F

One of the double angle formulas is $\sin(2x) = \frac{(1-\cos^2(x))}{2}$

Solution:
Wrong way.

3) T F

The function $f(x) = x(1-x^5) + \sin(\pi x)$ has a critical point in $(0,1)$.

Solution:
The derivative is positive at $x = 0$ and negative at $x = 1$ so that there is a point where.

4) T F

The function $f(x) = 1 + \sin(x^2) - x^3$ has a root in the interval $(-100, 100)$.

Solution:
Use the intermediate value theorem. The function satisfies $f(-100) \geq 100^3 - 2 > 0$ and $f(100) \leq -100^3 + 2 < 0$.

5) T F

To any continuous function f, there exists a unique $F(x)$ such that $F'(x) = f(x)$.

Solution:
F is only determined up to a constant. So, it is not unique.

6) T F If you listen to the sound $\ln(1+x)\sin(10000x)$, then it gets louder and louder as time goes on.

Solution:
The amplitude grows like $\ln(1+x)$. This is the hull function.

7) T F The function $f(x) = (x^{25}-1)/(x^5-1)$ has the limit 20 for $x \to 1$.

Solution:
Use Hospital's rule, or heal the function. The limit is $5 = 25/5$, not $20 = 25-5$.

8) T F If the average cost $F(x)/x$ of an entity is extremal at $x = 2$, then we have a break-even point $f(2) = g(2)$.

Solution:
This is the strawberry theorem.

9) T F The Midi function $f(s)$ gives the midi number $f(s)$ as a function of the frequency s.

Solution:
Wrong way. The variable s is the midi number and $f(s)$ is the frequency.

10) T F A Newton step for the function f is $T(x) = x + \frac{f(x)}{f'(x)}$.

Solution:
Wrong. The sign is off.

Problem 2) Algebra (10 points)

Simplify the following expressions until they become a quantity that does not depend on x any more.

a) $\frac{1}{\frac{1}{x}} - x$

b) $(1/x)/(x/x^2)$

c) $1 + \sqrt{x^6}x^{-3}$

d) $(2^x)^2 - (2^2)^x$

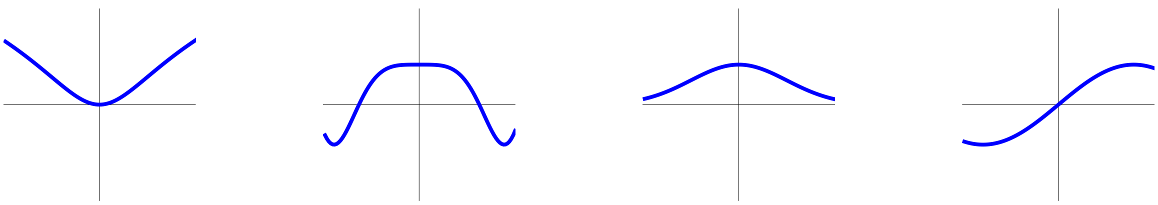
e) $\ln(e^{x+3}e^{-x})$.

Solution:
a) 0
b) 1
c) 2
d) 0
e) 3

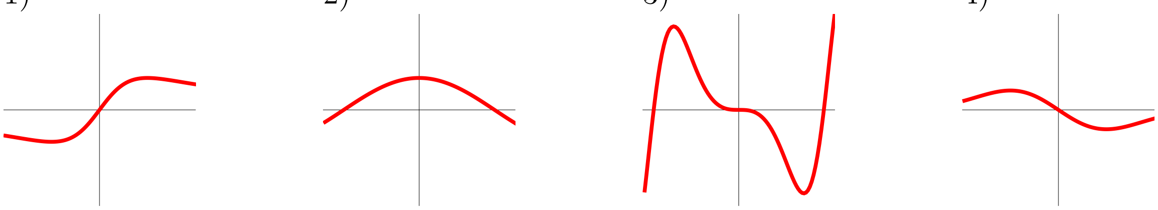
Problem 3) Functions (10 points)

a) (5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

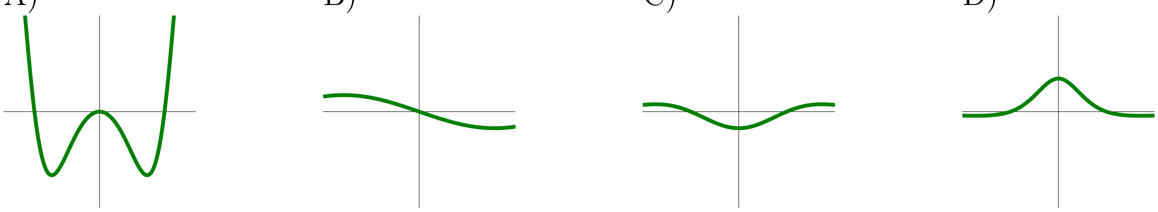
Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)$			
$\cos(x^2)$			
$\ln(1+x^2)$			
$\exp(-x^2)$			



1) 2) 3) 4)



A) B) C) D)



a) b) c) d)

Solution:
a) 4,B,b
2,C,a
1,A,d
3,D,c

Problem 4) Limits, Continuity (10 points)

Which of the following limits exist in the limit $x \to 1$? If the limit exists, enter the result:

Function	Enter the limit if it exists	Check if it does not exist
$\frac{(1-x^2)}{(1-x)}$		
$\frac{\ln\|1-x\|}{\ln(1+x)}$		
$\frac{(1-x^9)}{(1-x^7)}$		
$\frac{x}{\ln\|x\|}$		
$\frac{\tan(1-x)}{(1-x)}$		
$\ln\|x\|/x$		
$\ln(x)/\ln(2x)$		
$\frac{x^2-1}{\sin(x^2-1)}$		
$\frac{\cot(1-x)}{x}$		
$\frac{\text{sinc}(x)}{1-x}$		

Solution:
2, DNE, 9/7, DNE, 1,0,0,1, DNE, DNE

Problem 5) Related Rates (10 points)

a) (7 points) We know that

$$yx^3 + 2xy^3 + xy = 4$$

relates two functions $x = x(t), y = y(t)$ and that $x'(0) = 4$ and $x = x(0) = 1, y = y(0) = 1$. Find $y' = y'(0)$?

b) (3 points) For the same relation but if $y = y(x)$ we want to know $y'(x)$ at $x = 1$, $y = 1$.

Solution:
$y'x^3 + 3yx^2x' + 2y^3x' + 6xy^2y' + xy' + x'y = 0$. We can use this relation both for a) and b).
a) Fill in the points to get $y' + 12 + 8 + 6y' + y' + 4 = 0$. This gives $y' = -24/8 = -3$.
b) Filling the points gives now $y' + 3 + 2 + 7y' + 1 = 0$ giving $y' = -6/8 = -3/4$.

Problem 6) Integrals (10 points)

a) (5 points) Determine the value of the definite integral

$$\int_0^{\pi/2} e^{\sin(x)} \cos(x) \, dx \, .$$

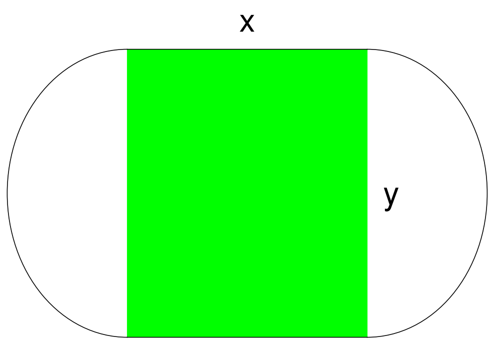
b) (5 points) Find the anti-derivative

$$\int (1+x)^3 + \frac{1}{1+x^2} + \frac{1}{\sqrt{1-x^2}} \, dx \, .$$

Solution:
a) $e - 1$.
b) $(1+x)^4/4 + \arctan(x) + \arcsin(x) + C$.

Problem 7) Extrema (10 points)

The Harvard stadium has a track which encloses a rectangular field of dimensions x, y. The circumference of the track is $400 = 2\pi y + 2x$ and is fixed. We want to maximize the area xy for a play field. Which x achieves this?



Solution:
Solve for $y = (200 - x)/\pi$ and plug this into the function to get

$$f(x) = xy = x(200 - x)/\pi \ .$$

To find the maximum of this function, we differentiate with respect to x and look where the derivative is zero:

$$f'(x) = (200 - 2x)/\pi = 0$$

showing that $x = 100$ is the maximum.

Problem 8) Substitution (10 points)

a) (5 points) **"One,Two,Three,Four Five, once I caught a fish alive!"**

$$\int \frac{(1+2x+3x^2+4x^3+5x^4)}{(1+x+x^2+x^3+x^4+x^5)}\, dx \ .$$

b) (5 points) A **"Trig Trick-or-Treat"** problem:

$$\int (1-x^2)^{-3/2} + (1-x^2)^{-1/2} + (1-x^2)^{1/2} \, dx \ .$$

Solution:
a) Substitute $u = 1 + x + x^2 + x^3 + x^4 + x^5$ so that we get $\int du/u = \ln(u) + c = \ln(1 + x + x^2 + x^3 + x^4 + x^5) + C$.
b) Use trig substitution $x = \sin(u)$ in all cases. We get

$$\int \frac{1}{\cos^2(u)} + 1 + \cos^2(u)\, du = \tan(u) + u + (1 + \sin(2u)/2)/2 + C$$

which is $\tan(\arcsin(x)) + \arcsin(x) + (1 + \sin(2\arcsin(x))/2)/2 + C$.

Problem 9) Integration by parts (10 points)

a) (5 points) Find

$$\int (1 + x + x^2 + x^3 + x^4)(\sin(x) + e^x) \, dx \, .$$

b) (5 points) Find

$$\int \ln(x) \frac{1}{x^2} \, dx \, .$$

Solution:
a) Use TicTacToe:

$1 + x + x^2 + x^3 + x^4$	$\sin(x) + e^x$	
$1 + 2x + 3x^2 + 4x^3$	$-\cos(x) + e^x$	$\oplus$
$2 + 6x + 12x^2$	$-\sin(x) + e^x$	$\ominus$
$6 + 24x$	$\cos(x) + e^x$	$\oplus$
24	$\sin(x) + e^x$	$\ominus$
0	$-\cos(x) + e^x$	$\oplus$

Collecting together, we could write $(4x^3 + 3x^2 - 22x - 5)\sin(x) + (x^4 - 3x^3 + 10x^2 - 19x + 20)e^x + (-x^4 - x^3 + 11x^2 + 5x - 23)\cos(x)$.
b) As we know from LIATE, we differentiate the log $\ln(x)$. We get

$$-\ln(x)\frac{1}{x} + \int_x \frac{1}{x^2} = -\ln(x)/x - 1/x + C \, .$$

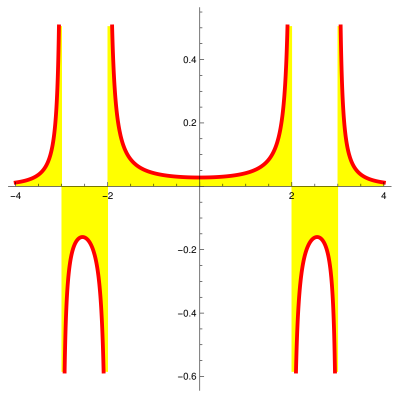
Problem 10) Fractions (10 points)

Integrate

$$\int_{-1}^{1} \frac{1}{(x+3)(x+2)(x-2)(x-3)} \, dx \; .$$

The graph of the function is shown to the right.

Lets call it the **friendship graph**.



Solution:
Use partial fraction with the Hospital method of course: Write

$$\frac{1}{(x+3)(x+2)(x-2)(x-3)} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{x-2} + \frac{D}{x-3}$$

To get A, multiply the entire equation with $x+3$, simplify and take the limit $x \to -3$. This gives $A = \frac{1}{(-3+2)(-3-2)(-3-3)} = -1/30$. Similarly, we get $B = 1/20$ and $C = -1/20$ and $D = 1/30$. Now we can write the integral as

$$\begin{aligned} & \frac{-1}{30} \ln|x+3| + \frac{1}{20} \ln|x+2| - \frac{1}{20} \ln|x-2| + \frac{1}{30} \ln|x-3| \Big|_{-1}^{1} \\ = & \frac{-1}{30}(\ln(4) - \ln(2)) + \frac{1}{20} \ln(3) - \frac{1}{20} \ln(3) + \frac{1}{30}(\ln(2) - \ln(4)) \; . \end{aligned}$$

Problem 11) Applications I. (10 points)

We look at two functions which are defined to be zero outside the interval $[0, \pi]$ and which are given on $0 \leq x \leq \pi$ as

$$f(x) = \frac{\sin(x)}{2}$$

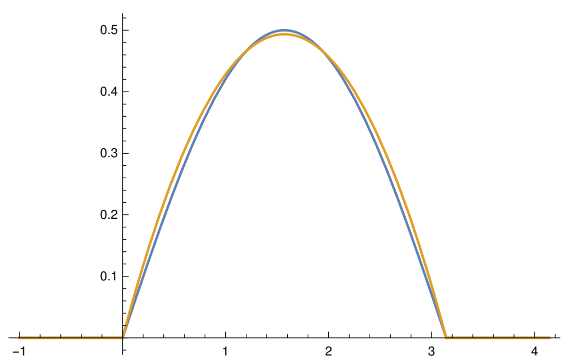
and

$$g(x) = \frac{x(\pi - x)}{5} .$$

Their graphs are very close to each other, as you can see in the picture.

a) (5 points) One of these functions produces a probability distribution. Which one? Justify your answer.

b) (5 points) The other function needs to be multiplied with a constant c to become a probability distribution. What is this constant?



Solution:
f is a probability distribution. It is non-negative and integrates up to 1: $\int_0^\pi \sin(x)/2 \; dsx = 1$. The second total integral for is $\pi^3/30$. If we multiply the function g with $30/\pi^3$ we get a probability density function.

Problem 12) Applications II (10 points)

a) (2 points) What is the expectation $\int x f(x)\, dx$ of the probability distribution function $f(x) = e^{-|x|}/2$ which is a function defined for all real x. [We only need a number.]

b) (2 points) What is the inverse function of the (fictional) **Stockhausen Midi function** $f(x) = 440 \cdot 5^{(x-25)/5}$? This function gives the midi number x as a function of the frequency f.

d) (2 points) The family of functions $f_c(x) = x^4 - cx^2 + x^2$ experiences a **catastrophe**. What is the parameter c for which this happens? [You only need to find that parameter.]

e) (2 points) If $(-2, -7)$ and $(2, 7)$ are two data points. What is the best linear fit $y = ax + b$ which passes through the two points?

Solution:
a) 0, $x = 25 + 5\ln_5(f/440)$, 9, $c = 1$, and $y = (7/2)x$.

Problem 13) Definitions (10 points)

a) (2 points) The logarithm is defined as the

of the exponential function.

b) (2 points) An expression of the form "0/0" or "∞/∞" is called

.

c) (2 points) If we simplify $f(x) = (x^2 - 1)/(x - 1) = x + 1$ we called this process also called

d) (2 points) If an function value is overall the largest on $[0, 1]$ it is called a [fill in the adjective]

maximum.

e) (2 points) A function F satisfying $F' = f$ is called

Solution:
a) inverse
b) indefinite form
c) healing
d) global max e) Anti derivative

Problem 14) Theorems (10 points)

a) State both forms of the fundamental theorem of calculus.

b) State the first and second derivative test.

c) State the intermediate value theorem .

d) State the mean value theorem.

e) State the extremal value theorem.

Solution:
a) $\int_0^x f'(t)\,dt = f(x) - f(0)$.
$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$.
b) First derivative test: If $f'(x)$ changes from positive to negative at x, then x is a local max if $f'(x)$ changes from negative to positive at x then x is a local min If $f'(x) = 0$ and $f'(x)$ does not change sign at $x = 0$, then x is neither a max nor a min Second derivative test: if $f'(x) = 0$ and $f''(x) > 0$ then x is a local min. c) If f is continuous and $f(a), f(b)$ have different signs, then there is $x \in (a, b)$ with $f(x) = 0$.
d) If f is differentiable on $[a, b]$, then there exists a point with $(f(b)-f(a))/(b-a) = f'(x)$.
e) If f is continuous on $[a, b]$, then there exists a point $x \in (a, b)$ where x is a global max and a point where x is a global min.