Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

5/8/2024: Final Exam Practice A

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

1		10
2		10
3		10
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7		10
8		10
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11		10
12		10
13		10
14		10
Total:		140

Problem 1) TF questions (10 points) No justifications are needed.

1) [T] [F] $e^{(x^y)} = e^{(y^x)}$ for all real numbers x, y.

2) [T] [F] There was a time in you life when your age $t \in \mathbb{R}$ in units of years was exactly your height $h \in \mathbb{R}$ in units in meters.

3) [T] [F] The function $f(x) = \sin(x)/x$ is called the entropy function.

4) [T] [F] Applying the Newton step with the function $f(x) = x^2 - 2$ at the point $x = 1$ gives $T(x) = 1 + 1/2 = 3/2$.

5) [T] [F] The family of functions $f_c(x) = c(x-1)^2 + 3$ experiences a catastrophe at $c = 0$.

6) [T] [F] If $F(x) = x^2 + 3x$ then $f(x) = 2x + 3$ is called marginal cost.

7) [T] [F] $\sin(3\pi/2) = -1$.

8) [T] [F] The cotangent function is monotonically decreasing on the open interval $(\pi/4, \pi/2)$.

9) [T] [F] If F is a CDF, then $F(0) = 0$.

10) [T] [F] If f is a PDF, then $\int_{-\infty}^{\infty} x^2 f(x)\, dx$ is called the variance of the distribution.

Problem 2) Algebra (10 points)

a) (5 points) Which of the following expressions are integers?

Expression	Integer (give value) or not?
$\arccos(\cos(2))$	
$e^{\ln(e^2)}$	
$\tan(\arctan(\tan(\pi/4)))$	
$e^{(e^0)}$	
$\ln(\ln(e))$	

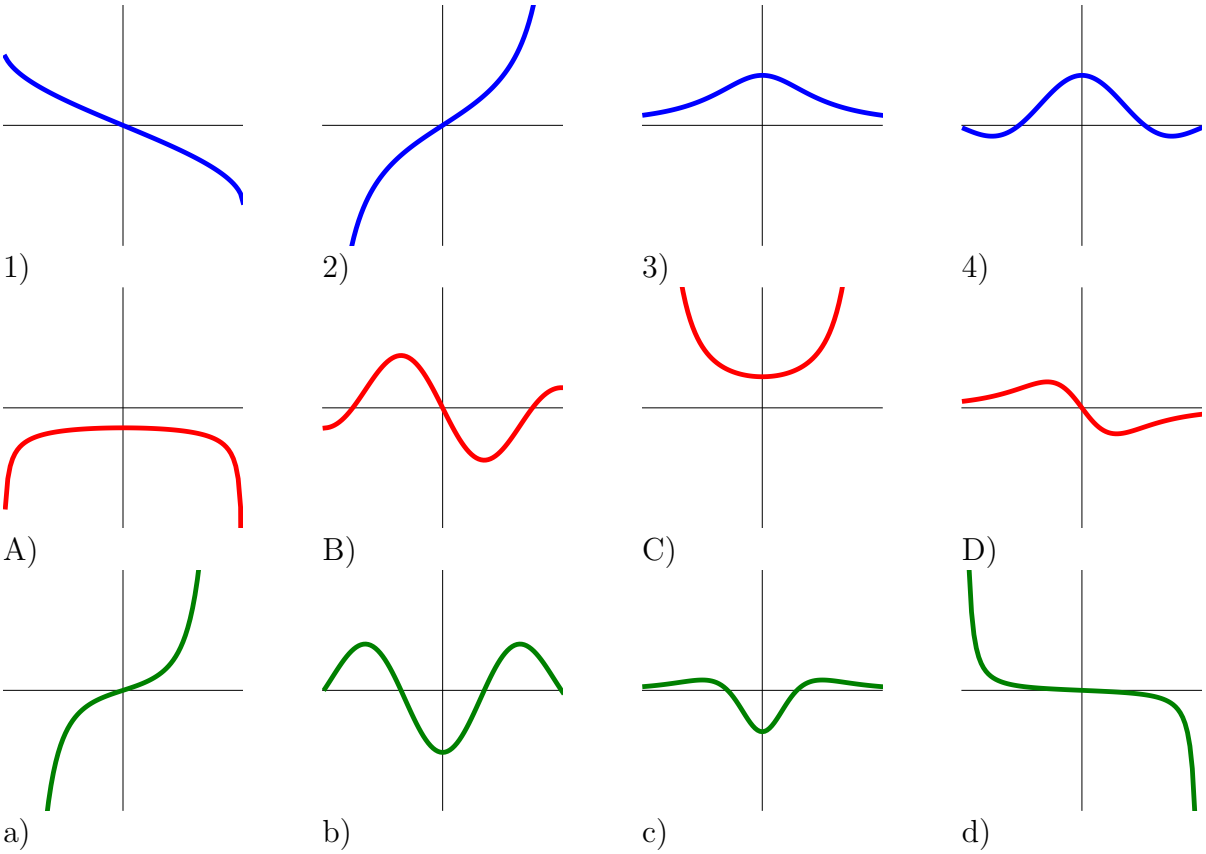
b) (5 points) Which of the following expressions are true for all x?

Expression	True or False
$\cos^2(2x) + \sin^2(2x) = 1$	
$\cos^2(2x) - \sin^2(2x) = \cos(2x)$	
$e^{e^{\ln(\ln(x))}} = x$	
$\sin(\arcsin(x)) = x$	
$\tan(\arctan(x)) = x$	

Problem 3) Functions (10 points)

Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)/x$			
$\tan(x)$			
$-\arcsin(x)$			
$1/(1+x^2)$			



Problem 4) Limits, Continuity (10 points)

a) (5 points) Which of the following limits exists in the limit $x \to 0$.

Function	exists	does not exist
$\sin^4(x)/x^4$		
$\arctan(x)/x$		
$\ln\|x\|/(x-1)$		
$\cos(x)/(x-1)$		
$(x^{10}-1)/(x-1)$		

b) (5 points) Describe in terms of the concept of "limit" what it means that a function $f(x)$

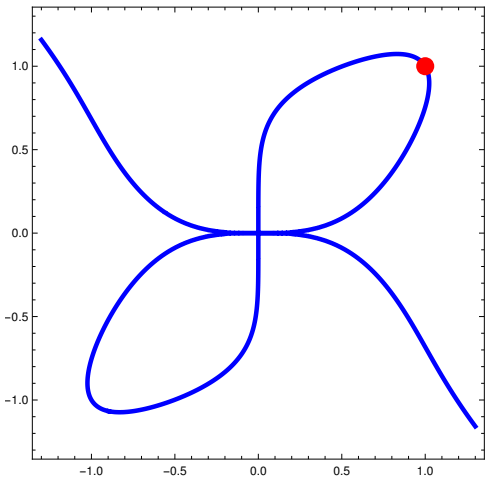
is continuous at 1	
is differentiable at 1	
has a jump discontinuity at 1	

Problem 5) Related Rates (10 points)

The coordinates of two cars on a freeway intersection are $x = x(t)$ and $y = y(t)$. We know

$$x^7 + y^7 - 2xy^2 = 0 \, .$$

Assume we know $x'(0) = 3$ and $x(0) = 1, y(0) = 1$. Find the derivative $y'(t)$.



Problem 6) Integrals (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = x^7 - \frac{1}{x}$

b) (2 points) $f(x) = x\ln(x) + \frac{1}{1+x^2}$.

c) (2 points) $f(x) = \frac{2x}{x^2+1} + \frac{1}{x^2-4}$.

d) (2 points) $f(x) = \sqrt{16 - x^2} + \frac{1}{\sqrt{1-x^2}}$.

e) (2 points) $f(x) = \ln(x) + \frac{1}{x\ln(x)}$.

Problem 7) Extrema (10 points)

A **trapezoid** of height $2x$, bottom length x and top length $(4 - 2x^2)$ has area

$$f(x) = x(x + (4 - 2x^2)) = 4x + x^2 - 2x^3 \ .$$

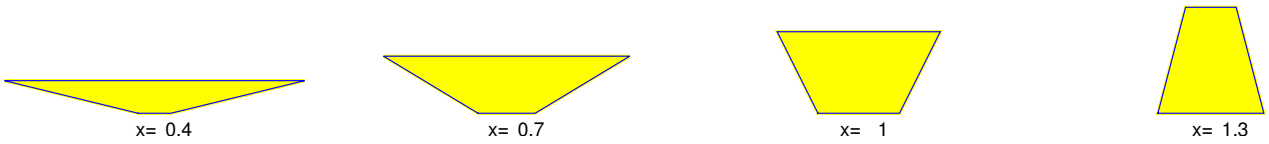
The figure below illustrates the situation for some x values.

a) (3 points) For which x does f have a local maximum? Use the second derivative test to check.

b) (3 points) On which interval $[a, b]$ does the problem make sense (height, lengths and area can not be negative)? What are the function values on the boundary?

c) (2 points) There a theorem which assures that $f(x)$ has a global maximum on that interval $[a, b]$? What is the name of the theorem?

d) (2 points) What is the global minimum of $f(x)$ on the interval of consideration?



Problem 8) Substitution (10 points)

a) (5 points)

$$\int \cos(\cos(\cos(x)))\sin(\cos(x))\sin(x)\ dx\ .$$

b) (5 points)

$$\int \frac{2\ln(\ln(\ln(x)))}{\ln(\ln(x))\ln(x)x}dx\ .$$

Problem 9) Integration by parts (10 points)

a) (5 points) Compute the following anti-derivative:

$$\int (x-2)^4 e^{x/2}\, dx\ .$$

b) (5 points) And now have some fun riding the "merry go round" for the following integral

$$\int e^x \sin(x)\, dx\ .$$

Problem 10) Fractions (10 points)

a) Integrate the definite integral $\int_5^6 \frac{4}{(x-3)(x+1)}\, dx$.

b) Integrate

$$\int \frac{1}{(x+8)(x+4)(x+2)}\, dx\ .$$

Problem 11) Applications I (10 points)

a) (2 points) Find the CDF $\int_0^x f(t)\ dt$ for the PDF which is $f(x) = \exp(-x/3)/3$ for $x \geq 0$ and 0 for $x < 0$.

b) (2 points) Perform a single Newton step for the function $f(x) = \sin(x)$ starting at $x = \pi/3$.

c) (2 points) Check whether the function $f(x) = 1/(2x^2)$ on the real line $(-\infty, \infty)$ is a probability density function.

d) (2 points) A rower produces the power $P(t)$ is $\sin^2(10t)$. Find the energy $\int_0^{2\pi} P(t)\ dt$ when rowing starting at time $t = 0$ and ending at $t = 2\pi$.

e) (2 points) What is the frequency of the Midi number 10 if you recall the formula $f = 4402^{(s-69)/12}$?

Problem 12) Applications II (10 points)

Lets look at a least square problem. Given the data points $(1, 2), (2, 3), (-3, -5)$, find the line $y = mx$ which best fits the data? You need to minimize

$$\sum_k (mx_k - y_k)^2$$

if (x_k, y_k) stands for the data points. Use one of the derivative theorems to verify that you got a minimum.

Problem 13) Definitions (10 points)

a) (2 points) What is the average rate of change of $f(x) = \sin(x)$ on the interval $[0, \pi/2]$.

b) (2 points) Why is the sigmoid function called logistic function ?

c) (2 points) If m is the mean of a probability distribution, then what is the name for $\int_{-\infty}^{\infty}(x-m)^2 f(x)\, dx$.

d) (2 points) What do we call the points for which $f(x) = 0$?

e) (2 points) What do we call the points for which $f'(x) = 0$?

Problem 14) Theorems (10 points)

The statement	is called the
A continuous function takes a maximum on $[0, 1]$	.
$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$	.
$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$	.
$\int_0^x f'(t)\, dt = f(x) - f(0)$	.
A wobbling table can be fixed by turning	
Break-even points are minima of the average cost	
If $f(0) = f(1) = 5$ there is $0 < x < 1$ with $f'(x) = 0$	
If $f(0) = 5$, $f(1) = 8$, there is $0 < x < 1$ with $f'(x) = 3$	
All functions can be approximated by neural nets	
If $f(0) = 4$ and $f(1) = -4$, then there is $0 < x < 1$ with $f(x) = 0$	