Intro

Unit 29

$$2\int_{-1}^{1} \sqrt{1-x^2}\,dx = ?$$

Try substitution.

$$\int \sqrt{1-x^2}\; 1\, dx$$

no!

$x = \sin u$, $dx = \cos u\, du$ works!

$$\int \frac{1+\cos 2u}{2}\, du$$

Practice

A $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\, dx$

B $\displaystyle\int (1-x^2)^{3/2}$

A $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx =$

$x = \sin u$

$dx = \cos u \, du$

$\sqrt{1-x^2} = \cos u$

$$= \int \frac{\cancel{\cos u}\,du}{\cancel{\cos u}} = \int 1 \, du$$

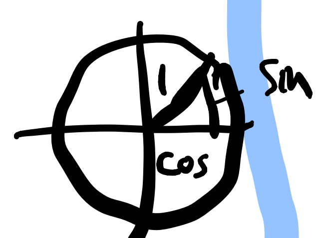
$$= u + c = \boxed{\arcsin(x) + c}$$

We know this already from the chain rule.

<u>Why:</u> $\sqrt{1-x^2}$

$$= \sqrt{1-\sin^2 u}$$

$$= \cos u$$



$$\boxed{\cos^2 u + \sin^2 u = 1}$$ Pythagoras

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13 $\displaystyle\int \frac{\sqrt{1-x^2}}{x}\,dx$

$x = \sin u$

$dx = \cos u \; du$

$$= \int \frac{\cos u}{\sin u} \cos u \, du$$

$\sqrt{1-x^2} = \sqrt{1-\sin^2 u}$

$= \sqrt{\cos^2 u}$

$= \cos u$

$$= \int \frac{\cos^2 u}{\sin u}\, du$$

$$= \int \frac{\sqrt{1-v^2}}{v}\, dv$$

$v = \sin u$

$dv = \cos u \, du$

$\cos u = \sqrt{1-v^2}$

Merry go round

We need more skills

→ Second part.

We need a $\boxed{\text{magic box}}$

→ below

$$\boxed{\text{Magic box}}$$

$$u = \tan\frac{x}{2}$$

$$dx = \frac{2\,du}{1+u^2}$$

$$\sin x = \frac{2u}{1+u^2}$$

$$\cos x = \frac{1-u^2}{1+u^2}$$

$$\boxed{\int \frac{1}{\cos x}\,dx}$$

$$\int \frac{\cancel{(1+u^2)}}{(1-u^2)}\,\frac{2\,du}{\cancel{(1+u^2)}}$$

residu method

$$= \int \frac{2\,du}{1-u^2} \qquad \frac{\cancel{(1-u)}}{(1+u)}$$

$$u=1$$

$$= \int \frac{A}{\cancel{1-u}}\cancel{(1-u)} + \frac{B}{1+u}(1+u)\,du$$

$$0$$

$$= \int \frac{1}{1-u}\,du + \frac{1}{1+u}\,du$$

$$= \boxed{-\log(1-u) + \log(1+u) + C}$$

and then
backsubstitute

$$x = \arctan(x) \cdot 2$$

Back to the
improvisation!

13 $\boxed{\int \frac{\cos^2 u}{\sin u}\, du}$

$$= \int \frac{(1-u^2)^2}{(1+u^2)^2} \frac{(1+u)^2}{2u} \cdot \frac{2du}{1+u^2}$$

$$= \int \frac{(1-u^2)^2}{(1+u^2)^2\, 2u}\, du$$

Partial fractions can solve this.

∫ am problems

A

$$\int \sqrt{1-x^2}\, dx$$

$$x = \cos u$$
$$dx = -\sin u \, du$$

$$= \int -\sin^2 u \, du$$

$$= \int \frac{\cos 2u - 1}{2}\, du$$

$$= \boxed{\frac{\sin 4u}{4} - \frac{u}{2}} + c$$

$$= \boxed{\frac{\sin 4 \arcsin x}{4} - \frac{\arcsin x}{2}} + c$$

B $\displaystyle\int \frac{\arccos x}{\sqrt{1-x^2}}\, dx$

=

$x = \cos u$

$u = \arccos x$

$\sqrt{1-x^2} = \sin u$

$dx = -\sin u\, du$

$$= \int u \, du$$

$$= \frac{u^2}{2} + C$$

$$= \boxed{\frac{\arcsin^2 x}{2} + C}$$

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5) $$\int \frac{\tan \frac{x}{2}}{\sin x} \, dx$$

$$= \int u \frac{2\,du}{(1+u^2)} \frac{(1+u^2)}{2u}$$

$$= \int 1 \, du = u + C$$

$$= \boxed{\tan\frac{x}{2} + C}$$