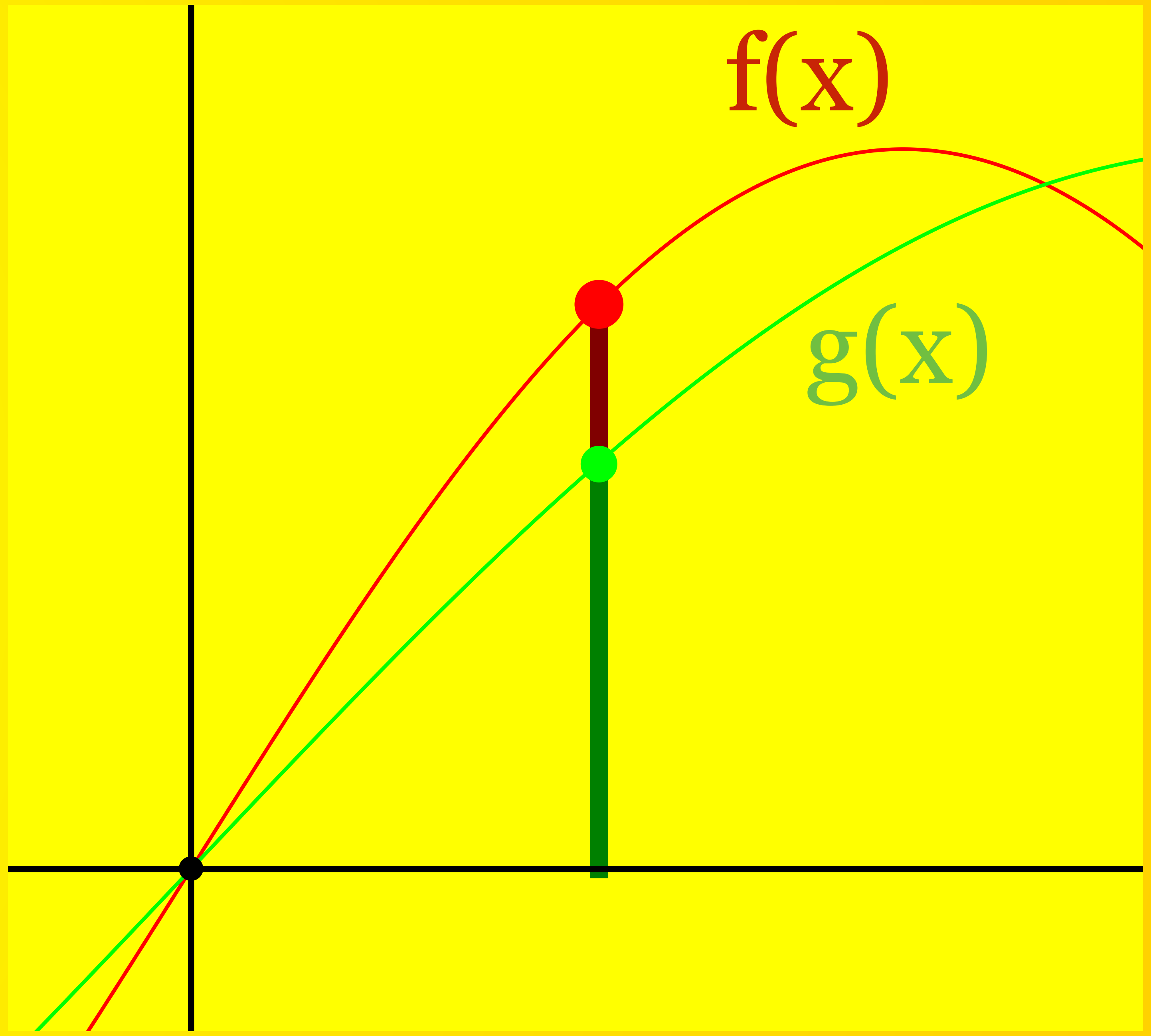
13 L'Hospital Rule

Hospital rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

assuming that f(a)=g(a)=0.

or that f(a)=g(a)=∞.



Example:

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin(3x)} = \lim_{x\to 0} \frac{5\cos(5x)}{3\cos(3x)} = \frac{5}{3}$$

PLAN

1. Poll

2. The rule

3.Indeterminate forms

4. Why?

5.Examples

6. JAM

POLL

What is

$$\lim_{x \to 0} \frac{\sin(7x)}{5x}$$

A 0

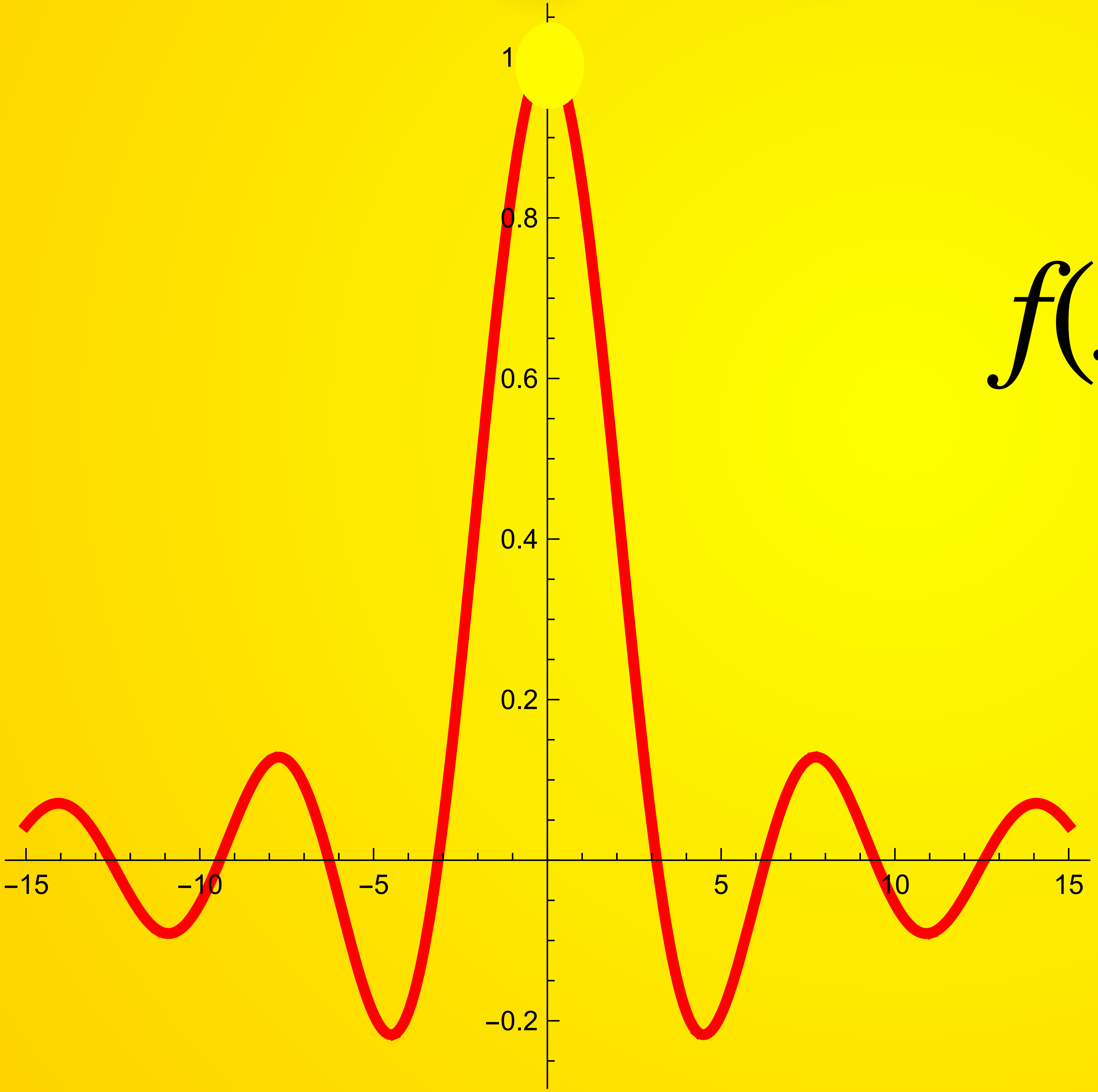
B ∞

C 1

D 5/7

E 7/5

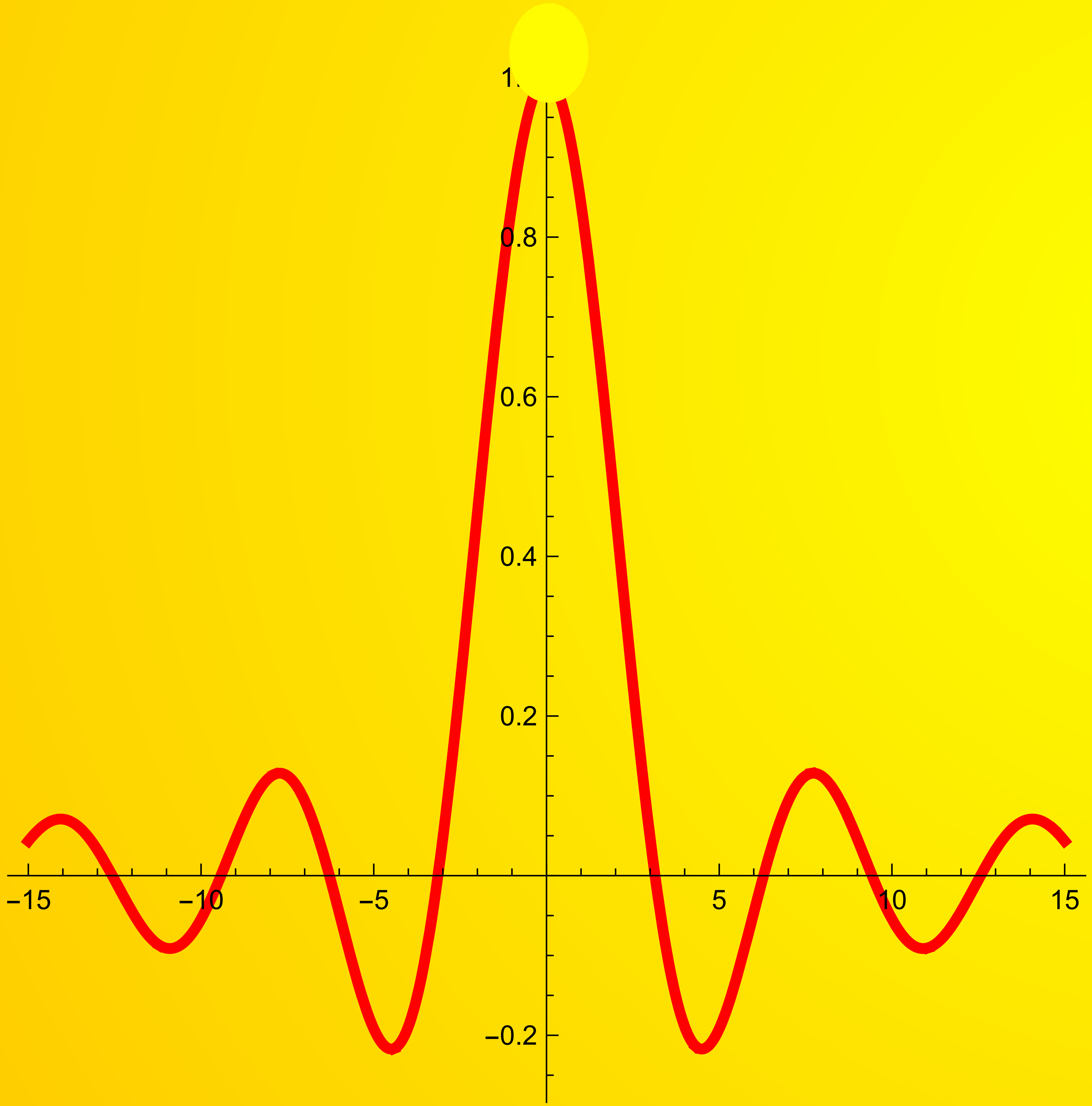
BROKEN BONES



$$f(x) = \frac{\sin x}{x}$$

What to do?

HOSPITAL



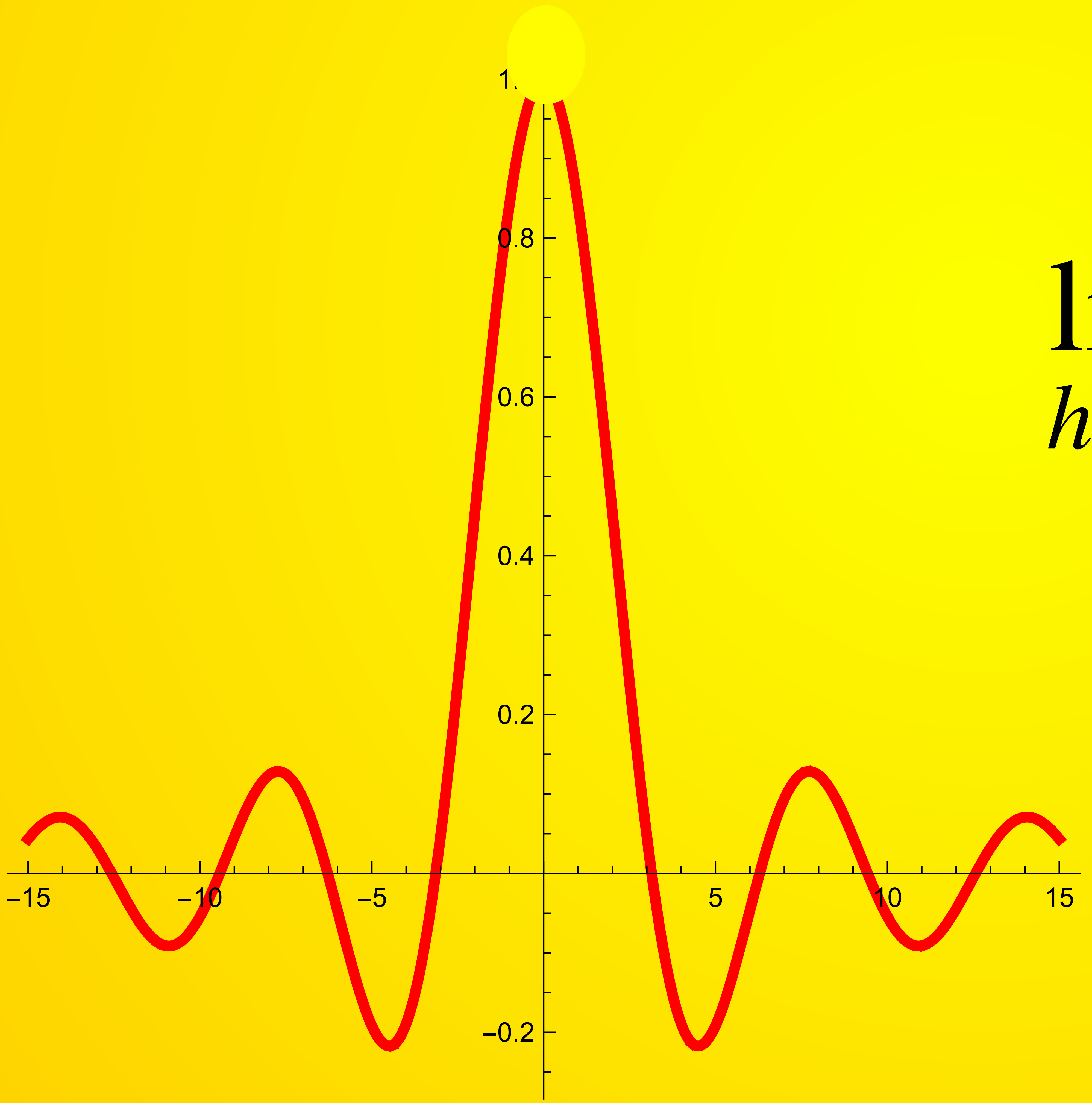
Bring it to the hospital!

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos(x)}{1}$$

$$=1$$


The wrong Missy, 2020

WHY?



$$\lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h-0}$$

WHY?

$$\frac{f'(a)}{g'(a)} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{g(a+h)-g(a)}$$

$$\lim_{h\to 0}\frac{f(a+h)-0}{g(a+h)-0}$$

f(a)=0

f(b)=0

$$\lim_{h\to 0}\frac{f(a+h)}{g(a+h)} = \lim_{x\to a}\frac{f(x)}{g(x)}$$

INDETERMINATE FORMS

Abbe Francois Moigno

1804-1884

An expression in which the result depends on how the limits are taken is called an indefinite form

$$\frac{\infty}{\infty}$$

$$\infty - \infty$$

$$0 \cdot \infty$$

$$0^{\infty} \qquad \infty^{0}$$

$$\frac{0}{0} \qquad 0^{0}$$

$$1^{\infty} = e^{0 \cdot \infty}$$

EXAMPLES

A $$\lim_{x\to 0} \frac{\tan(5x)}{x}$$

B $$\lim_{x\to 0} x\log(x)$$

C $$\lim_{x\to \infty} \frac{\log(x)}{x}$$

D $$\lim_{x\to 0} \frac{e^x - 1}{e^{3x} - 1}$$

E $$\lim_{x\to 0} \frac{1 - \cos(x)}{x^2}$$

F $$\lim_{x\to 0} \frac{e^{x^2} - 1}{e^x - 1}$$

TOUGHER

$$\lim_{x\to 0^+} \sqrt{x}\log(x)$$

$$\lim_{x\to\infty} x/\log(x)$$

$$\lim_{x\to -\infty} x^2 e^x$$

a $\lim_{x\to\pi} \frac{\sin(x)}{(x-\pi)(x+3)}$

b $\lim_{x\to -3} \frac{\sin(x)}{(x-\pi)(x+3)}$

c $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

d $\lim_{x\to\infty} \frac{\log(x)}{x^{1/3}}$

e $\lim_{x\to 3^-} \frac{\log(9-x^2)}{\arcsin(x-3)}$

f $\lim_{x\to\infty} \frac{\sin(x)}{x}$

g $\lim_{x\to 0} \frac{x}{\arccos(x)}$

IN CASE WE HAVE TIME...

JAM

$$\lim_{x\to\infty} \frac{e^{-3x}}{e^{-5x}}$$

$$\lim_{x\to 1} \frac{x}{\tan(x)}$$

$$\lim_{x\to 0} \frac{e^{-3x}}{e^{-5x}}$$

$$\lim_{x\to 1} \frac{\log(x)}{x-1}$$

$$\lim_{x\to 0} \frac{\cos(3x)-1}{\cos(7x)-1}$$

$$\lim_{x\to 0} \frac{x}{\tan(x)}$$

HISTORY

Guillaume de l'Hopital

Johann Bernoulli

The origin of L'Hôpital's rule

by D. J. Struik, Massachusetts Institute of Technology, Cambridge, Massachusetts

The so-called rule of L'Hôpital, which states that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

when $f(a)=g(a)=0$, $g'(a)\neq 0$, was published for the first time by the French mathematician G. F. A. de l'Hôpital (or De Lhospital) in his *Analyse des infiniment petits* (Paris, 1696) [1].* The Marquis de l'Hôpital was an amateur mathematician who had become deeply interested in the new calculus presented to the learned world by Leibniz in two short papers, one of 1684 and the other of 1686. Not quite convinced that he could master the new and exciting branch of mathematics all by himself, L'Hôpital engaged, during some months of 1691–92, the services of the brilliant young Swiss physician and mathematician, Johann Bernoulli, first at his Paris home and later at his château in the

* Numerals in brackets refer to the notes at the end of this article.

Several letters from Bernoulli to his patron with answers to questions have now been published, and the one dated July 22, 1694, contains the rule for $\frac{0}{0}$. The formulation is very much like the one we find in the *Analyse des infiniment petits*, and is based on a geometrical consideration. In our words, if

$$y=\frac{f(x)}{g(x)}$$

and both curves $y=f(x)$ and $y=g(x)$ pass through the same point P on the x axis, $OP=a$, so that $f(a)=g(a)=0$, and if we take an ordinate $x=a+h$, then the figure shows immediately that

$$\frac{f(a+h)}{g(a+h)}$$

is almost equal to the quotient of $hf'(a+h)$ and $hg'(a+h)$ when h is small. In the limit we find, now in Bernoulli's words:

"In order to find the value of the ordinate (*appliquée*) of the given curve

$$\left[y=\frac{f(x)}{g(x)}\right]$$

in this case it is necessary to divide the differential (*la différentielle*) of the numerator of the general fraction by the differential of the denominator."

HISTORY

Bernoulli's examples are almost the same that L'Hôpital uses:

1) $y=\dfrac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$ for $x=a$. Then

$y=\left(\dfrac{16}{9}\right)a$. This example is used by both Bernoulli and L'Hôpital.

The End