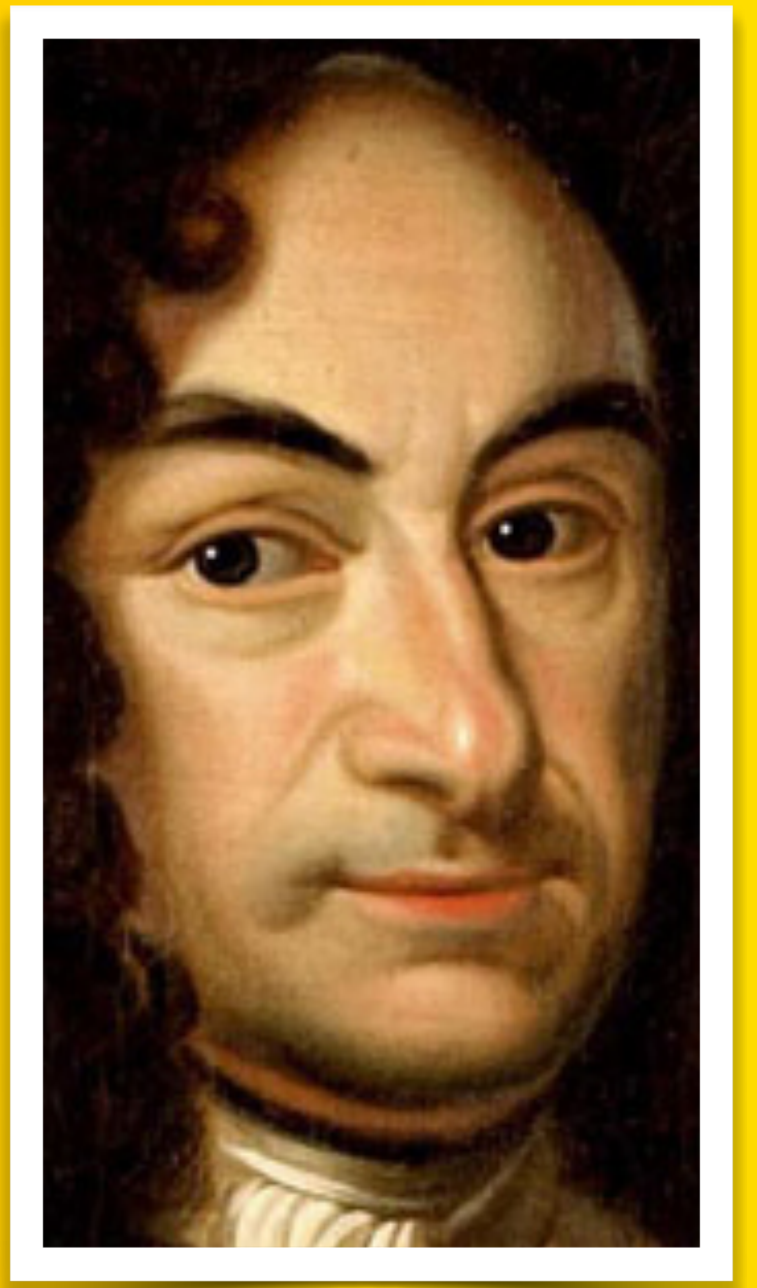
9

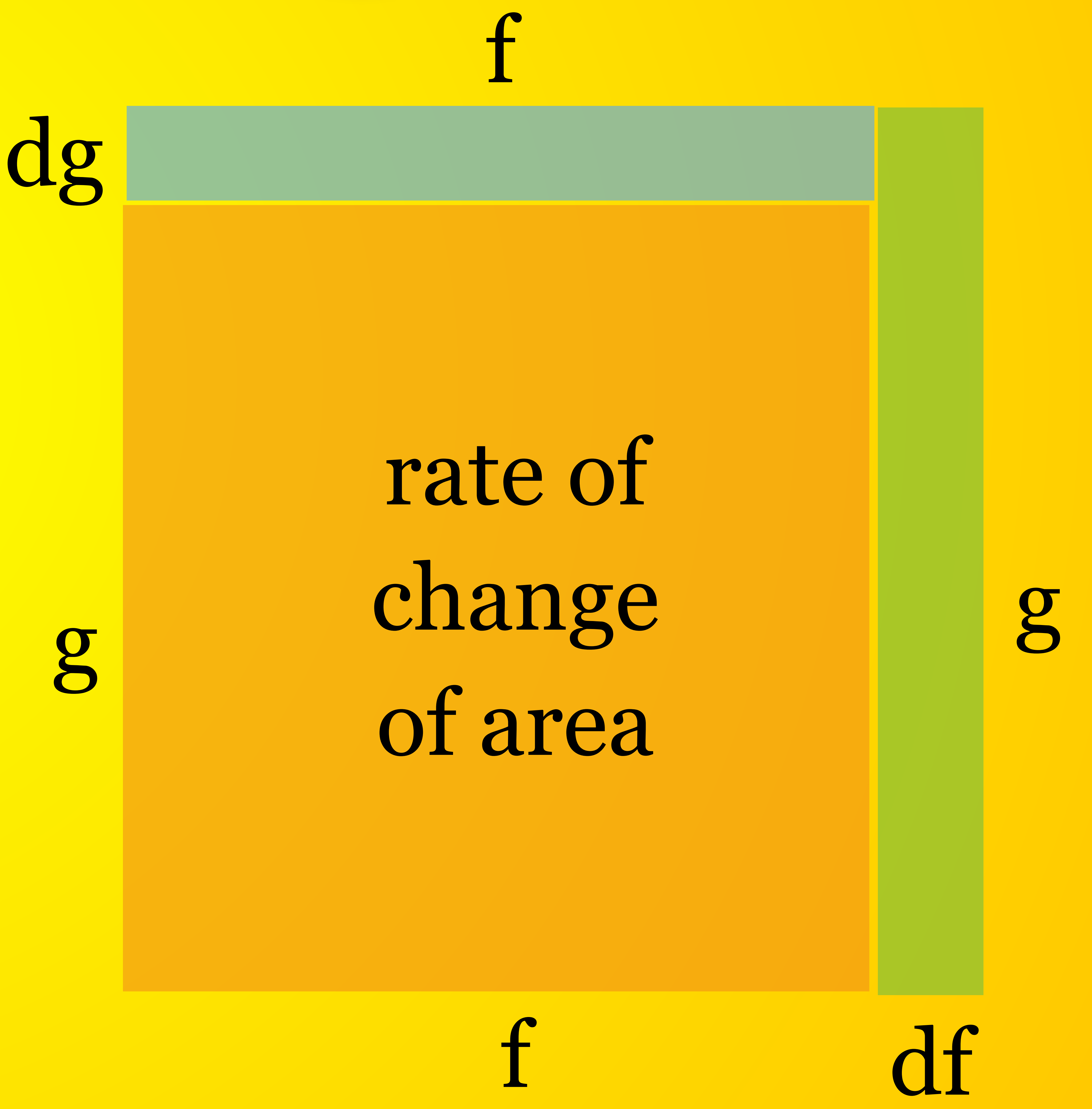
Product Rule



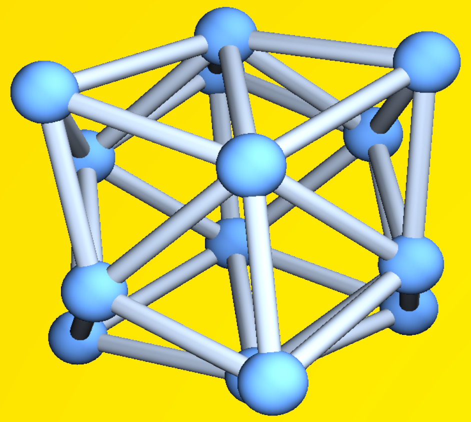
$(f\,g)' = f'\,g + f\,g'$

$(1/f)' = -f'/f^2$

$(f/g)' = (g\,f' - f\,g')/g^2$



PLAN



0. Project Reminder!

1. Poll

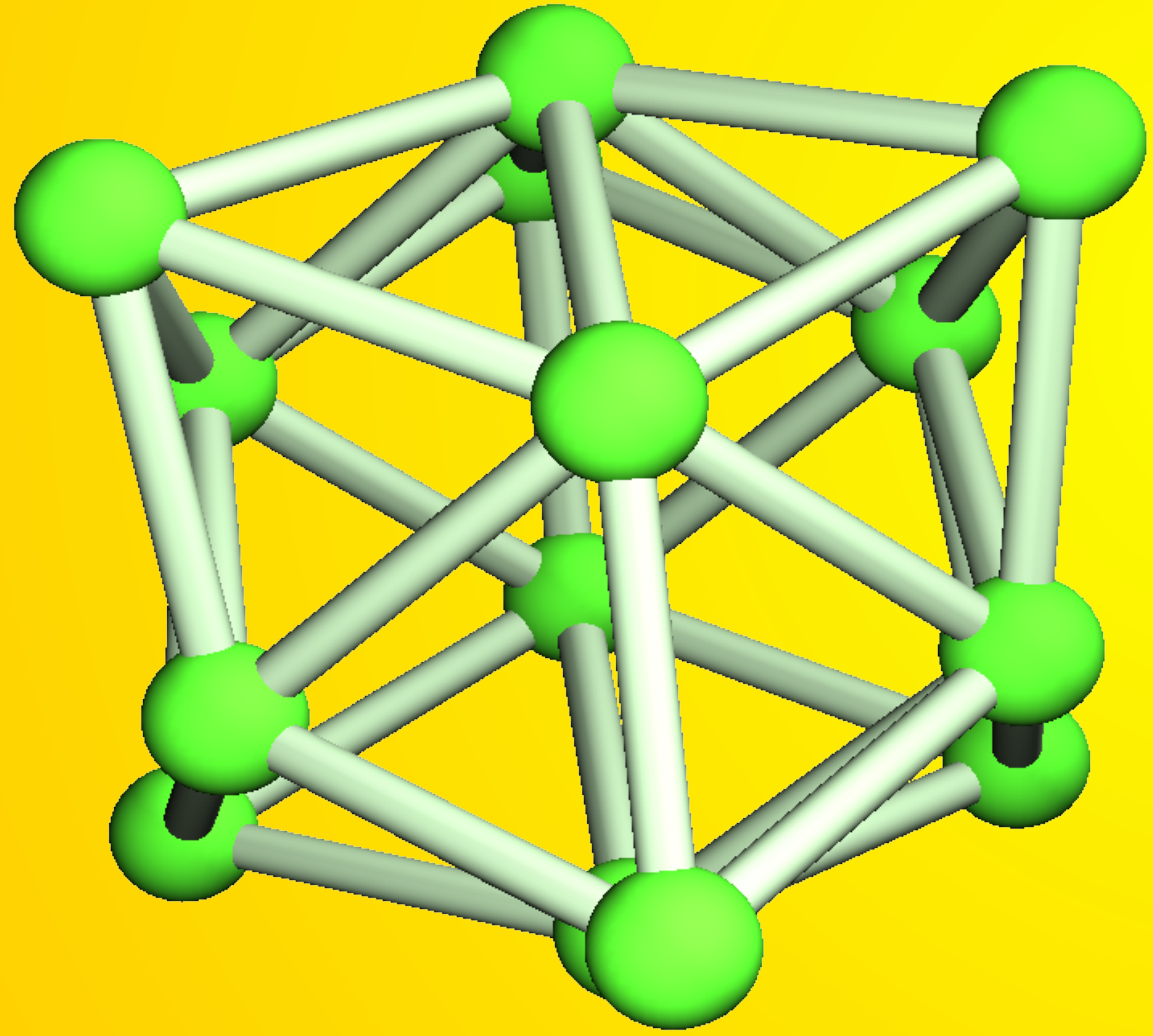
2. Product rule

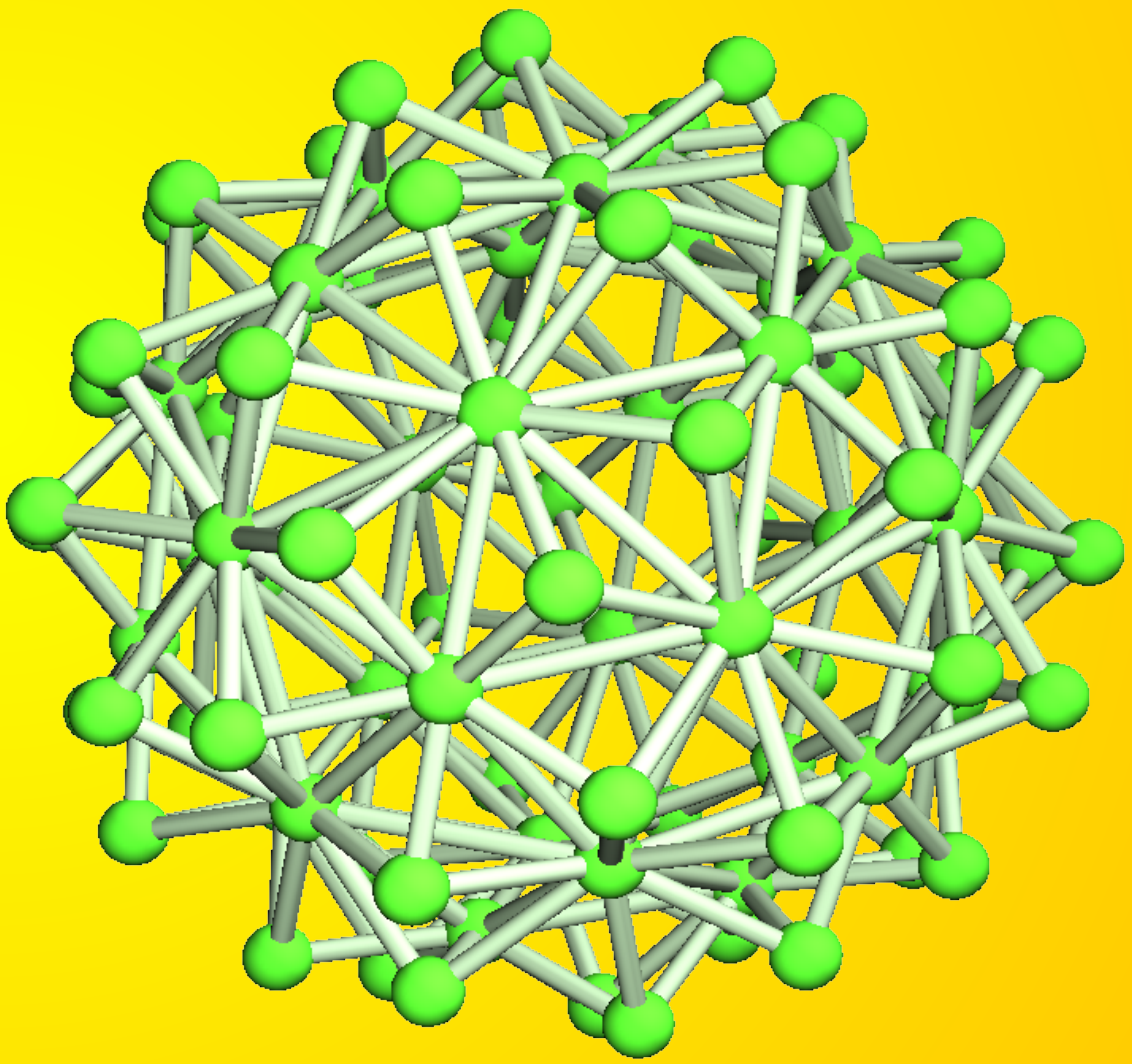
2. Reciprocal rule

3. Quotient rule

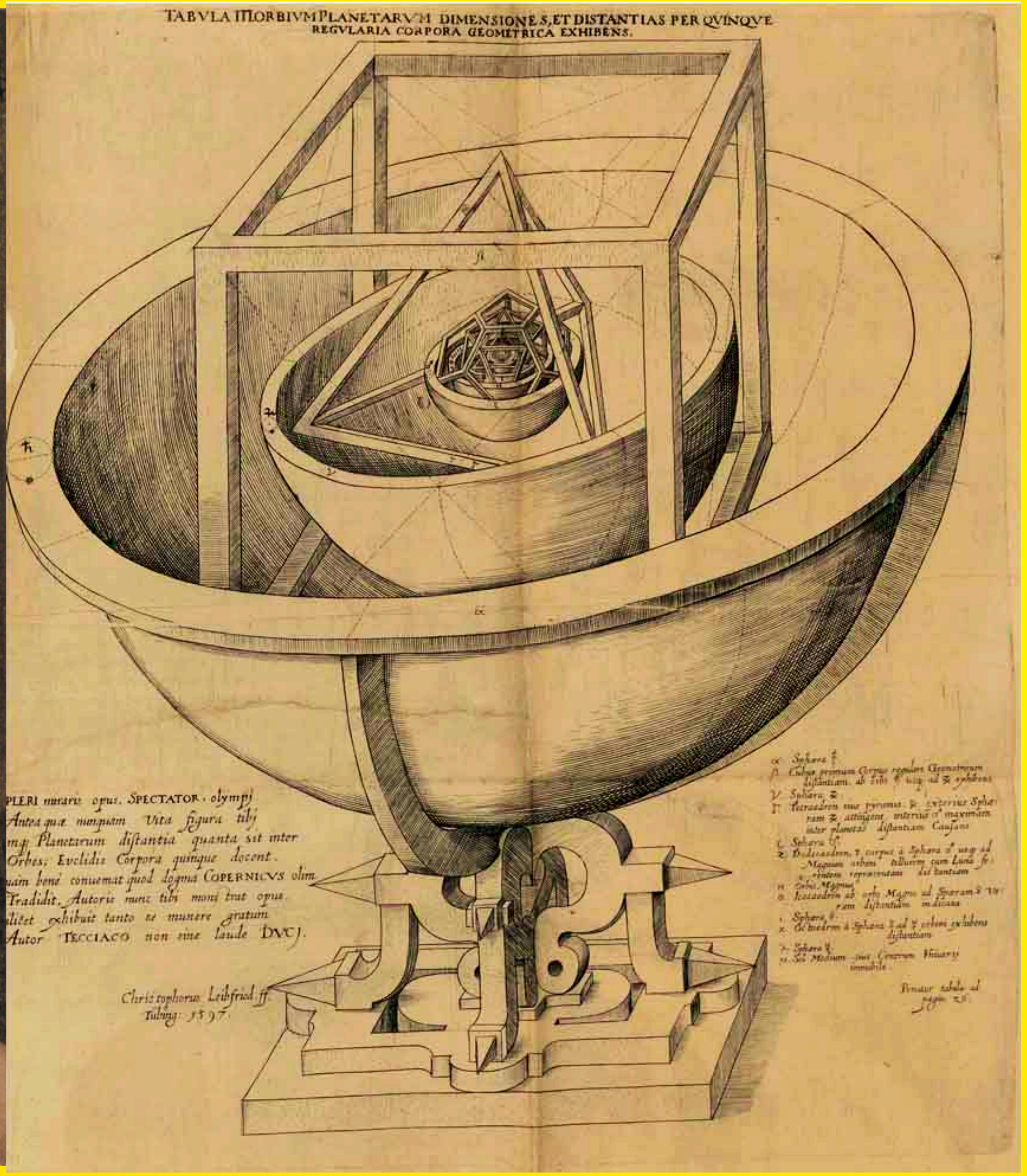
4. Examples

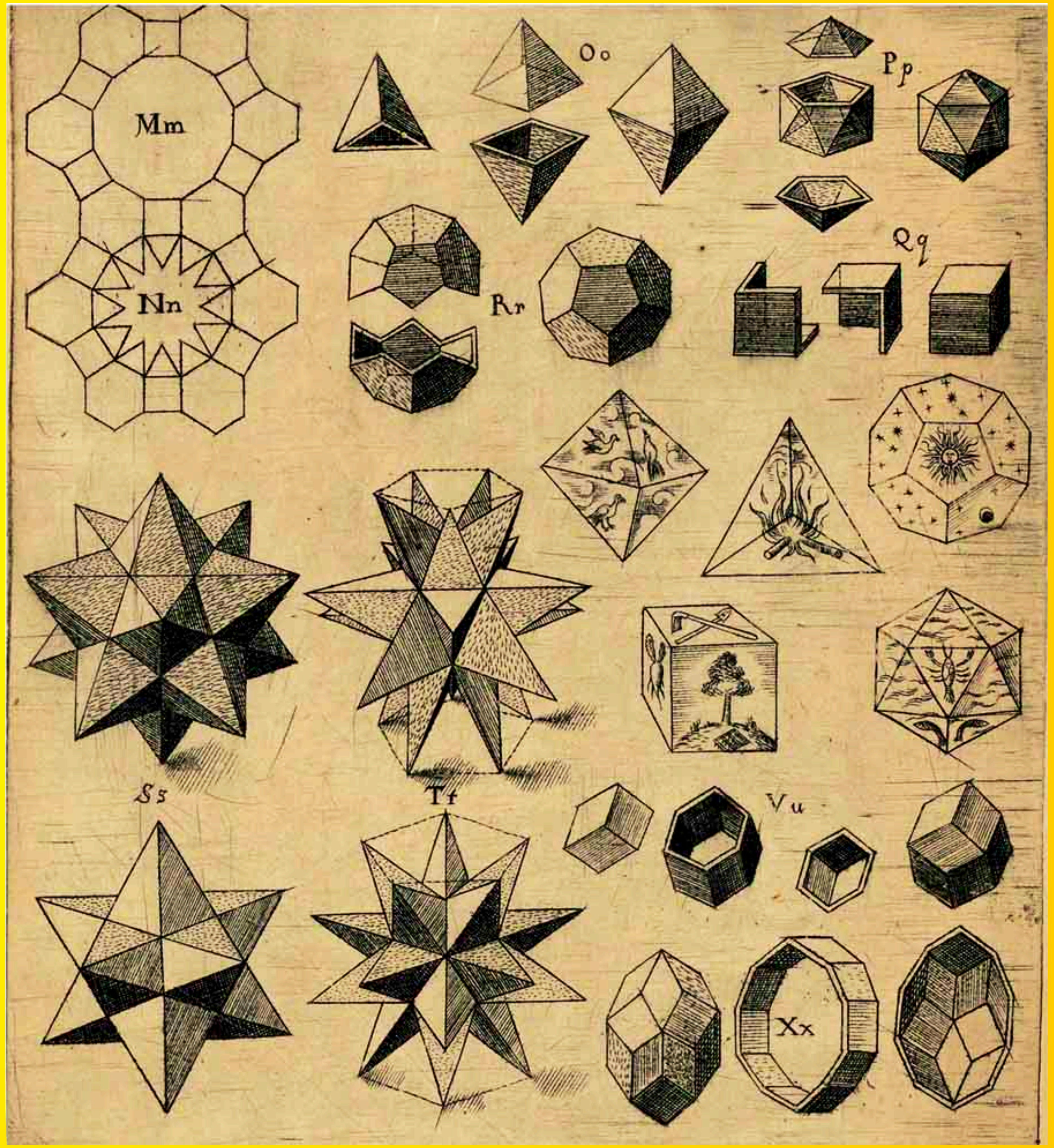
PROJECT





HARMONICES MUNDI





Kepler, 1618

NETWORKS

POLL

What is

$$\lim_{x \to 1} \frac{(x^2 - 1)(x - 1)}{(x^2 - 2x + 1)}$$

A. Not defined

B. 2

C. 1/2

D. 1

POLL

What is

$$\lim_{h \to 0} \frac{\sin(\pi + h) - \sin(\pi))}{h}$$

A. Not defined

B. 1

C. -1

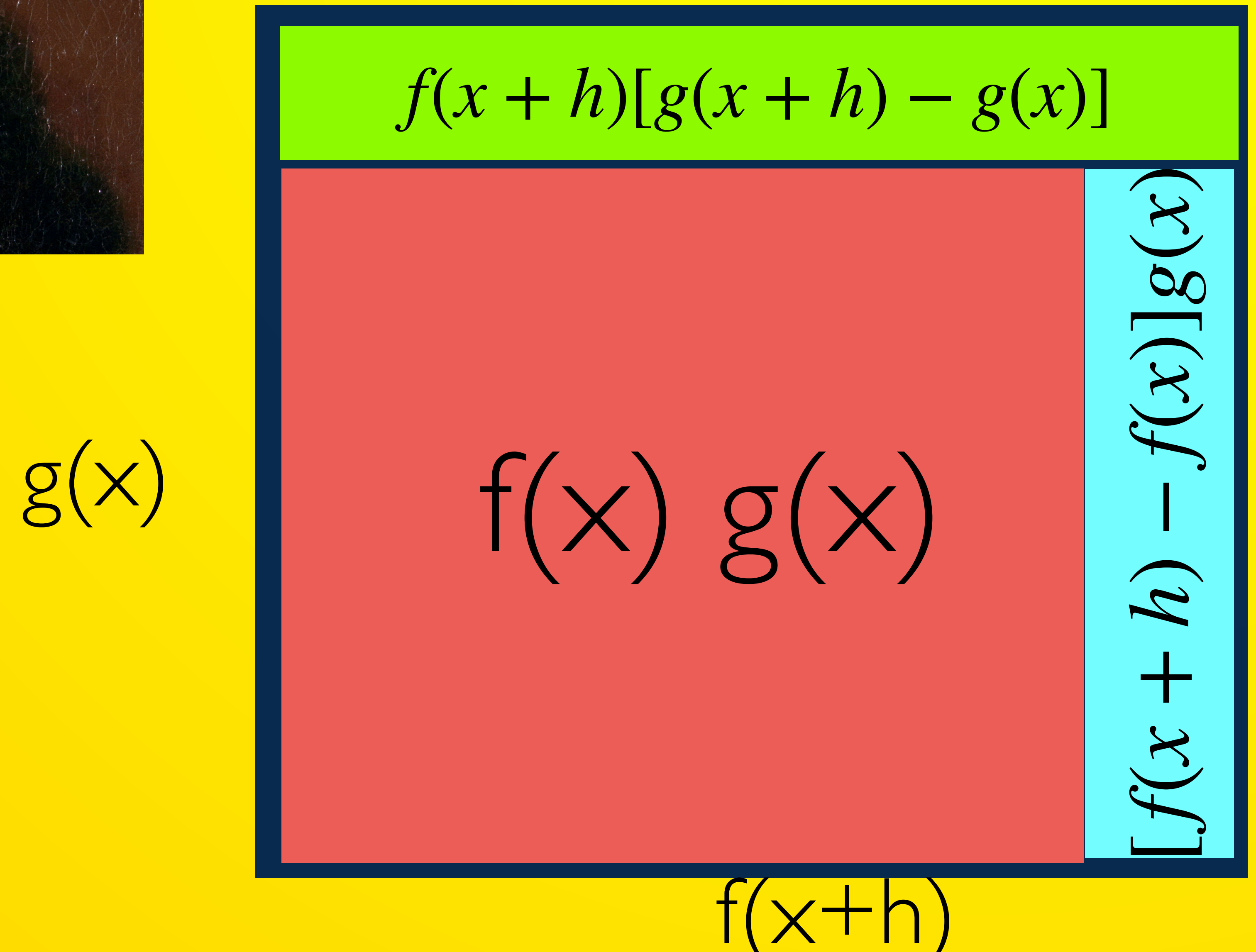
D. 0

PRODUCT RULE

1646-1716

$$(fg)' = f'g + fg'$$

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \frac{[f(x+h) - f(x)]g(x)}{h} + \frac{f(x+h)[g(x+h) - g(x)]}{h}$$



Now divide by h

Taking limits $h \to 0$ gives the product rule

SOURCE

Leibniz found it 1675-

Ordinatæ, ad axem normales, VX, WX, YX, ZX, quæ vocentur reſpective, v, w, y, z; & ipſa AX abſciſſa ab axe, vocetur x. Tangentes ſint VB, WC, YD, ZE axi occurrentes reſpective in punctis B, C, D, E. Jam recta aliqua pro arbitrio aſſumta vocetur dx, & recta quæ ſit ad dx, ut v (vel w, vel y, vel z) eſt ad VB (vel WC, vel YD, vel ZE) vocetur dv (vel dw, vel dy vel dz) ſive differentia ipſarum v (vel ipſarum w, aut y, aut z) His poſitis calculi regulæ erunt tales:

Sit a quantitas data conſtans, erit da æqualis 0, & d $\overline{ax}$ erit æqu. a dx: ſi ſit y æqu v (ſeu ordinata quævis curvæ YY, æqualis cuivis ordinatæ reſpondenti curvæ VV) erit dy æqu. dv. Jam *Additio & Subtractio*: ſi ſit z − y + w + x æqu. v, erit d $\overline{z - y + w + x}$ ſeu dv, æqu dz − dy + dw + dx. *Multiplicatio*, d $\overline{xv}$ æqu. x dv + v dx, ſeu poſito y æqu. xv, fiet dy æqu x dv + v dx. In arbitrio enim eſt vel formulam, ut xv, vel compendio pro ea literam, ut y, adhibere. Notandum & x & dx eodem modo in hoc calculo tractari, ut y & dy, vel aliam literam indeterminatam cum ſua differentiali. Notandum etiam non dari ſemper regreſſum a differentiali Æquatione, niſi cum quadam cautione, de quo alibi. Porro *Diviſio*, d $\frac{v}{y}$ vel (poſito z æqu. $\frac{v}{y}$) dz æqu. $\frac{\pm v\,dy \mp y\,dv}{yy}$

Quoad *Signa* hoc probe notandum, cum in calculo pro litera ſubſtituitur ſimpliciter ejus differentialis, ſervari quidem eadem ſigna,

DISCOVERY

Leibniz found it 1675-1677

11 November, 1673.[26]

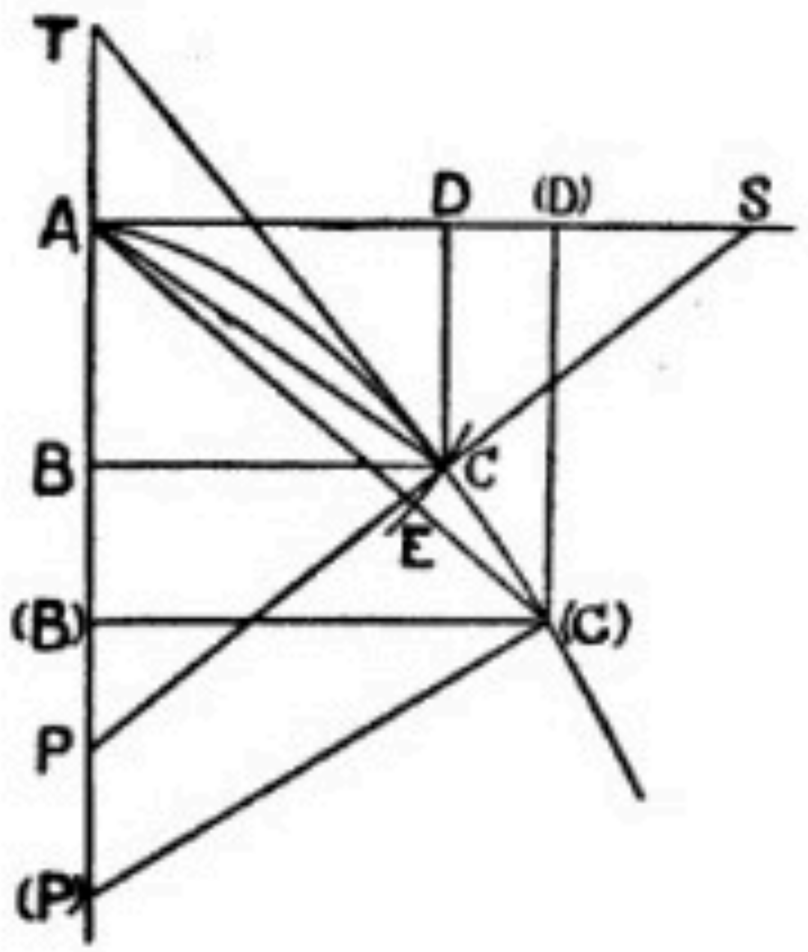
Methodi tangentium inversae exempla.

[Examples of the inverse method of tangents.]

A year or two ago I asked myself the question, what can be considered one of the most difficult things in the whole of geometry, or, in other words, what was there for which the ordinary methods had contributed nothing profitable. To-day I found the answer to it, and I now give the analysis of it.

Find the curve C(C), *in which* BP, *the interval between the ordinate* BC *and* PC *the normal to the curve, taken along the axis* AB(B), *is reciprocally proportional to the ordinate* BC.

Let AD(D) be another straight line perpendicular to the axis AB(B), and let ordinates CD be drawn to it, so that the abscissae AD along the axis AD(D) are equal to the ordinates BC to the axis AB(B), and the ordinates CD to the axis AD(D) are equal to the abscissae AB along the axis AB(B). Let us call AD=BC=y, and AD=BC=x; also let BP=w and B(B)=z. Then it follows from what I have proved in another place that



$$\int wz = \frac{y^2}{2}, \text{ or } wz = \frac{y^2}{2d} \quad [27]$$

But from the quadrature of a triangle it is evident that $\frac{y^2}{2d} = y$; and therefore $wz = y$.

[26] See Cantor, III, p. 183; but neither Cantor nor Gerhardt appears to offer any suggestion as to why this date should have been altered.

[27] See foot of next page.

Leibniz took our course!

Leibniz 1680

fore also a circle will be found equal to the surface of the parabolic conoid; but this is not the place to deduce it at full length.

Now these, which may seem to be great matters, are only the very simplest results to be obtained by this calculus; for many much more important consequences follow from it, nor does there occur any simple problem in geometry, either pure or applied to mechanics, that can altogether evade its power. Now we will expound the elements of the calculus itself.

The fundamental principle of the calculus.

Differences and sums are the inverses of one another, that is to say, the sum of the differences of a series is a term of the series, and the difference of the sums of a series is a term of the series; and I enunciate the former thus, $\int dx = x$, and the latter thus, $d\int x = x$.

Thus, let the differences of a series, the series itself, and the sums of the series, be, let us say,

Diffs.		1	2	3	4	5	....	dx
Series	0	1	3	6	10	15	....	x
Sums		0	1	4	10	20	25 ..	$\int x$

Then the terms of the series are the sums of the differences, or $x = \int dx$; thus, $3 = 1 + 2$, $6 = 1 + 2 + 3$, etc.; on the other hand, the differences of the sums of the series are terms of the series, or $d\int x = x$; thus, 3 is the difference between 1 and 4, 6 between 4 and 10.

Also $da = 0$, if it is given that a is a constant quantity, since $a - a = 0$.

Addition and Subtraction.

The difference or sum of a series, of which the general term is made up of the general terms of other series by addition or subtraction, is made up in exactly the same manner from the differences or sums of these series; or

$$x + y - v = \int \overline{dx + dy - dv}, \quad \int \overline{x + y - v} = \int x + \int y - \int v.$$

This is evident at sight, if you take any three series, set out their sums and their differences, and take them together correspondingly as above.

Simple Multiplication.

Here $dxy = x dx + y dy$, or $xy = \int x dx + \int y dy$.

This is what we said above about figures taken together with their complements being equal to the circumscribed rectangle. It is demonstrated by the calculus as follows:

dxy is the same thing as the difference between two successive xy's; let one of these be xy, and the other $x + dx$ into $y + dy$; then we have

$$dxy = \overline{x + dx} \,.\, \overline{y + dy} - xy = x\,dy + y\,dx + dx\,dy;$$

the omission of the quantity $dx\,dy$, which is infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small (because the lines are understood to be continuously increasing or decreasing by very small increments throughout the series of terms), will leave $x\,dy + y\,dx$; the signs vary according as y and x increase together, or one increases as the other decreases; this point must be noted.

Simple Division.

Here we have $d\dfrac{y}{x} = \dfrac{x\,dy - y\,dx}{xx}$.

For, $d\dfrac{y}{x} = \dfrac{y + dy}{x + dx} - \dfrac{y}{x} = \dfrac{x\,dy - y\,dx}{xx + x\,dx}$, which becomes (if we write xx for $xx + x\,dx$, since $x\,dx$ can be omitted as being infinitely small in comparison with xx) equal to $\dfrac{x\,dy - y\,dx}{xx}$; also, if $y = aa$, then $dy = 0$, and the result becomes $-\dfrac{aadx}{xx}$, which is the value we used a little while before in the case of the tangent to the hyperbola.

From this any one can deduce by the calculus the rules for *Compound Multiplication and Division*; thus,

$$dxvy = xy\,dv + xv\,dy + yv\,dx,$$

$$d\frac{y}{vz} = \frac{vz\,dy - yv\,dz - yz\,dv}{vv.zz};$$

as can be proved from what has gone before; for we have

$$d\frac{y}{x} = \frac{x\,dy - y\,dx}{xx};$$

hence, putting zv for x, and $z\,dv + v\,dz$ for dx or dzv in the above, we obtain what was stated.

EXAMPLES

$$\frac{d}{dx}x^5x^7$$

$$\frac{d}{dx}e^{2x}e^{3x}$$

$$\frac{d}{dx}\sin(x)x^5$$

EXAMPLES

$$\frac{d}{dx}x\log(x)$$

$$\frac{d}{dx}e^{x}e^{3x}$$

$$\frac{d}{dx}(x^4+1)^2$$

$$\frac{d}{dx}e^{-7x}x^2$$

RECIPROCAL RULE

$$(1/f)' = \frac{-f'}{f^2}$$

The product rule gives

$$0 = 1' = (f\frac{1}{f})' = f'(1/f)'$$

Now solve for (1/f)'

EXAMPLES

$$\frac{d}{dx}\sec(x)$$

$$\frac{d}{dx}1/x^5$$

$$\frac{d}{dx}1/e^x$$

$$\frac{d}{dx}\csc(x)$$

QUOTIENT RULE

$$(f/g)' = (f'g - fg')/g^2$$

1646-1716

The product rule gives

$$f' = \left(g\frac{f}{g}\right)' = g'\frac{f}{g} + g\left(\frac{f}{g}\right)'$$

$$f'g = g'f + g^2\left(\frac{f}{g}\right)'$$

Now solve for (f/g)'

$dz - dy + dw + dx$. *Multiplicatio*, $d\,\overline{xv}$ æqu. $x\,dv + v\,dx$, seu posito y æqu. xv, fiet dy æqu $x\,dv + v\,dx$. In arbitrio enim est vel formulam, ut xv, vel compendio pro ea literam, ut y, adhibere. Notandum & x & dx eodem modo in hoc calculo tractari, ut y & dy, vel aliam literam indeterminatam cum sua differentiali. Notandum etiam non dari semper regressum a differentiali Æquatione, nisi cum quadam cautione, de quo alibi. Porro *Divisio*, $d\frac{v}{y}$ vel (posito z æqu. $\frac{v}{y}$) dz æqu. $\frac{\pm v\,dy \mp y\,dv}{yy}$

Quoad *Signa* hoc probe notandum, cum in calculo pro litera substituitur simpliciter ejus differentialis, servari quidem eadem signa, & pro $+z$ scribi $+dz$, pro $-z$ scribi $-dz$, ut ex additione & subtractione paulo ante posita apparet; sed quando ad exegesin valorum venitur, seu cum consideratur ipsius z relatio ad x, tunc apparere, an valor ipsius dz sit quantitas affirmativa, an nihilo minor seu negativa: quod posterius cum fit, tunc tangens ZE ducitur a puncto Z non versus A, sed in partes contrarias seu infra X id est tunc cum

EXAMPLES

$$\frac{d}{dx}\frac{x^2+x}{x^4+1}$$

$$\frac{d}{dx}\frac{x^4}{x^6}$$

$$\frac{d}{dx}\frac{1}{\log(x)}$$

$$\frac{d}{dx}\frac{x^4}{\frac{\sin(x)}{\log(x)}}$$

The Song

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Low Di High
Take High Di Low
Cross the line
and square the low

JAM

$$\frac{d}{dx}\tan(x)$$

$$\frac{d}{dx}\sec x\csc(x)$$

$$\frac{d}{dx}\sin^2(x)$$

$$\frac{d}{dx}x\sin(x)$$

$$\frac{d}{dx}\exp(x)\sin(x)\cos(x)\log(x)$$

$$\frac{d}{dx}\sqrt{\sin(x)}$$

(*) without chain rule

The End