4/9/2021: Second hourly, Practice E

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.

- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) **T** F   The formula $\int_0^x f''(x) \, dx = f'(x) - f'(0)$ holds.

Solution:
Apply the fundamental theorem to the derivative.

2) **T** F   The area of the upper half disc is the integral $\int_{-1}^{1} \sqrt{1 - x^2} \, dx$

Solution:
The circle has the equation $x^2 + y^2 = 1$. Solving for y gives $y = \sqrt{1 - x^2}$.

3) **T** F   If the graph of the function $f(x) = x^2$ is rotated around the interval $[0, 1]$ in the x axes we obtain a solid with volume $\int_0^1 \pi x^4 \, dx$.

Solution:
Yes, the cross section area is $A(x) = \pi x^4$. Integrate this from 0 to 1.

4) T **F**   The function $f(x) = e^x$ is the only anti derivative of e^x.

Solution:
We can add a constant c.

5) **T** F   If f has a critical point at 1, then $F(x) = \int_0^x f(t) \, dt$ has an inflection point at 1.

Solution:
By the fundamental theorem of calculus, $F' = f$ and $F'' = f'$. If f has the critical point 1, then $f'(1) = 0$ and so $F''(1) = 0$ which by definition means that F has an inflection point. An inflection point for F is a point, where $F'' = f'$ changes sign. That means $f' = 0$.

6) **T** F   Catastrophes are parameter values c for a family of functions $f_c(x)$, for which a local minimum of f_c disappears.

7) [T] [F] The volume of a cylinder of height and radius 1 minus the volume of a cone of height and radius 1 is half the volume of a sphere of radius 1.

Solution:
This was Archimedes insight and you have discovered yourself in a homework.

8) [T] [F] The function $f(x) = 1/(\pi\sqrt{1 - x^2})$ for $-1 < x < 1$ and $f(x) = 0$ else is a PDF

Solution:
It is the arc sin solution.

Solution:
There could also be critical points outside the interval.

9) [T] [F] The improper integral $\int_0^1 x^{1/3} \, dx$ is finite.

Solution:
Yes

10) [T] [F] Integrals are linear: $\int_0^x f(t) + g(t) \, dt = \int_0^x f(t) \, dt + \int_0^x g(t) \, dt$.

Solution:
You have verified this in a homework.

11) [T] [F] The function $\text{Li}(x) = \int_2^x dt/\log(t)$ has an anti-derivative which is a finite construct of trig functions.

Solution:
No, it is known that this logarithmic integral has no elementary anti-derivative.

12) **T** F
There is a region enclosed by the graphs of x^5 and x^6 which is finite and positive.

Solution:
Yes, there is a finite region enclosed. It is between $x = 0$ and $x = 1$.

13) T **F**
The integral $\int_{-1}^{1} 1/x^4 \ dx = -1/(5x^5)|_{-1}^{1} = -1/5 - 1/5 = -2/5$ is defined and negative.

Solution:
It is not defined at 0 and can not even be saved using the Cauchy principal value.

14) **T** F
Gabriel's trumpet has finite volume but infinite surface area.

Solution:
Yes, we have seen this in class.

15) **T** F
A function $f(x)$ is a probability density function, if $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x) \ dx = 1$.

Solution:
This is the definition of a probability density function.

16) T **F**
The mean of a function on an interval $[a, b]$ is $\int_a^b f(x) \ dx$.

Solution:
For the mean, we would have to divide by $(b - a)$.

17) **T** F
The cumulative probability density function is an anti-derivative of the probability density function.

Solution:
Yes, by definition. It is a particular anti-derivative which has the property that $\lim_{x \to -\infty} F(x) = 0$.

18) ☐ T ☒ F   The integral $\int_{-\infty}^{\infty}(x^2 - 1)\,dx$ is finite.

Solution:
No way. The function does not even go to zero at infinity. $\int_{-R}^{R}(x^2-1)\,dx = (x^3/3-x)|_{-R}^{R} = 2R^3/3 - 2R$ does not converge for $R \to \infty$.

19) ☐ T ☒ F   The total prize is the derivative of the marginal prize.

Solution:
Its reversed. The marginal prize is the derivative of the total prize.

20) ☐ T ☒ F   The acceleration is the anti-derivative of the velocity.

Solution:
It is reversed. The acceleration is the derivative of the velocity.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their anti derivatives. Of course only 6 of the 30 functions will appear.

Function	Anti-derivative Enter 1-30
$\cos(3x)$	
$\sin(3x)$	
$3x$	

Function	Anti-derivative Enter 1-30
$1/(3x)$	
$\tan(3x)$	
$1/(1+9x^2)$	

1) $\sin(3x)$
2) $-\sin(3x)/3$
3) $\sin(3x)/3$
4) $-3\sin(3x)$
5) $3\sin(3x)$

6) $\cos(3x)$
7) $-\cos(3x)/3$
8) $\cos(3x)/3$
9) $-3\cos(3x)$
10) $3\cos(3x)$

11) $\log(x)/3$
12) $1/(3-x)$
13) $1/(3x)$
14) $\log(x/3)$
15) $-1/(3x^2)$

16) $3x^2$
17) $x^2/2$
18) $3x^2/2$
19) 3
20) x^2

21) $\arctan(3x)/3$
22) $3\arctan(3x)$
23) $1/(1+9x^2)$
24) $3/(1+9x^2)$
25) $-3/(1+x^2)$

26) $1/\cos^2(3x)$
27) $\log(\cos(3x))$
28) $-\log(\cos(3x))/3$
29) $\log(\cos(3x))/3$
30) $3/\cos^3(3x)$

Solution:
The magic numbers are $3, 7, 18, 11, 28, 21$.

Problem 3) Matching problem (10 points) No justifications are needed.

Which of the following formulations is a Riemann sum approximating the integral $\int_0^3 f(x)\,dx$ of $f(x) = x^2$ over the interval $0, 3$.

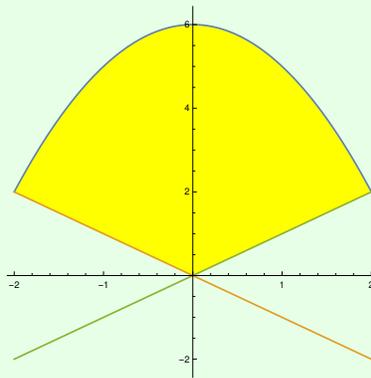| Sum	Check if this is the Riemann sum.
$n\sum_{k=0}^{n-1}(3k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{n-1}(3k/n)^2$	
$n\sum_{k=0}^{3n-1}(k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{3n-1}(k/n)^2$	

Problem 4) Area computation (10 points)

Find the area of the region enclosed by the three curves $y = 6 - x^2$, $y = -x$ and $y = x$ which is above the x axes.

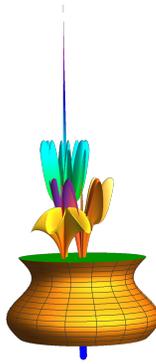**Solution:**
Make a picture:

We can look at the region on the right of the y-axis and have therefore the area between the graph of $f(x) = x$ and $f(x) = 6 - x^2$. These graphs intersect at $x = 2$. The integral is

$$\int_0^2 (6 - x^2 - x) \, dx = 22/3 \ .$$

Since we have only computed half, we have to multiply this with 2. The area is 44/3.

Problem 5) Volume computation (10 points)

Emma Woodhouse grows plants in a pot which is a rotationally symmetric solid for which the radius at position x is $5 + \sin(x)$ and $0 \le x \le 2\pi$. Find the volume of the pot.

Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x^2 \sin(\pi x) \, dx$.

b) (5 points) $\int_1^3 x^2 \cos(x^3 + 2) \, dx$

Problem 7) Anti-derivatives (10 points)

Find the following anti-derivatives

a) (5 points) $\int \frac{3}{\sqrt{1-x^2}} + x^4 + \frac{1}{1+x^2} \, dx$

b) (5 points) $\int \frac{1}{(x-2)(x+4)} + \frac{2}{x-1} \, dx$

Problem 8) Chain rule and volume (10 points)

A coffee machine has a filter which is a cone of radius z at height z. If the coffee covers everything until height t, let $V(t)$ denote the volume and $z(t)$ the height at time t. Coffee spills out at a rate of $V'(t) = 1$ cubic centimeter per second. How fast does the water level sink at height $z = 10$? To solve this, note that the chain rule equates $d/dt V(t)$ with $\pi z^2 z'(t)$.
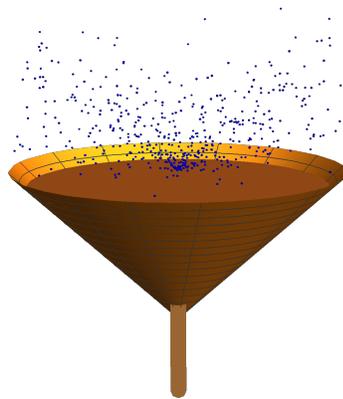
Problem 9) PDF's and CDF's (10 points)

Assume we know that the CDF is given by $f(x) = 1/2 + \arctan(x)/\pi$.
Determine the corresponding PDF.

Solution:
$\frac{1}{\pi}\frac{1}{1+x^2}$.

Problem 10) Improper integrals (10 points)

Evaluate the following improper integrals or state that they do not exist

a) (3 points) $\int_1^\infty 1/\sqrt{x}\,dx$.

b) (2 points) $\int_0^1 \sqrt{x}\,dx$.

c) (3 points) $\int_0^\infty 2xe^{-x^2}\,dx$.

d) (2 points) $\int_0^\infty \frac{1}{x}\,dx$

Solution:
a) $\int_1^R 1/\sqrt{x}\,dx = 2\sqrt{x}|_1^R$ does not have a limit for $R \to \infty$.
b) $\int_0^1 \sqrt{x}\,dx = x^{3/2}(2/3)|_0^1 = 2/3$.
c) Use substitution $-e^{-x^2}|_0^\infty = 1$.
d) $\log(x)|_a^R = \log(R) - \log(a)$ does not exist on both ends. For $a \to 0$ and for $R \to \infty$.