Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

4/9/2021: Second hourly, Practice D

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] [F] The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

Solution:
Differentiate the right hand side to check.

2) [T] [F] The fundamental theorem of calculus implies that $\int_0^1 f'(x)\ dx = f(1) - f(0)$.

Solution:
Yes this is a special case of the fundamental theorem.

3) [T] [F] The volume of truncated pyramid with a base square length 2 and top square length 3 is given by the integral $\int_2^3 x^2\ dx$.

Solution:
Yes the area of a slice is x^2.

4) [T] [F] The derivative of $\arctan(x)$ is $1/\cos^2(x)$.

Solution:
The derivative of $\tan(x)$ is $1/\cos(x)^2$.

5) [T] [F] The mean value theorem implies $\int_a^b f'(x)\ dx = f'(c)(b-a)$ for some c in the interval (a, b).

Solution:
This is a typical application of the mean value theorem.

6) [T] [F] If $F(x) = \int_0^x f(t)\ dt$ has an critical point at $x = 1$ then f has a root at $x = 1$.

Solution:
The first derivative of F is f.

7) [T] [F] The anti-derivative of the derivative of f is equal to $f + C$ where C is a constant.

Solution:
This is a consequence of the fundamental theorem.

8) [T] [F] The CDF is an anti-derivative of a PDF.

Solution:
By definition

9) [T] [F] The identity $\frac{d}{dx}\int_5^9 f(x)\,dx = f(9) - f(5)$ holds for all continuous functions f.

Solution:
We differentiate a constant.

10) [T] [F] Two surfaces of revolution which have the same cross section area $A(x)$ also have the same volume.

Solution:
This is Archimedes insight and true.

11) [T] [F] The integral $\int_{-\infty}^0 e^x\,dx$ is an improper integral which converges.

Solution:
Yes, we integrate over an infinite interval and can compute that the integral is 1.

12) [T] [F] The identity $\int_2^9 7f(x)\,dx = 7\int_2^9 f(x)\,dx$ is true for all continuous functions f.

Solution:
Yes, we can take the 7 constant outside the integral.

13) [T] [F] The improper integral $\int_1^\infty 1/x\ dx$ in the sense that $\int_1^R 1/x\ dx$ converges for $R \to \infty$ to a finite value.

Solution:
This is what we mean with the existence. But the integral does not exist.

14) [T] [F] If $f_c(x)$ has a local minimum at $x = 2$ for $c < 1$ and no local minimum anywhere for $c > 1$, then $c = 1$ is a catastrophe.

Solution:
This is a definition.

15) [T] [F] An improper integral is an indefinite integral which does not converge.

Solution:
These two terms are easy to mix up. Improper means that we either have a discontinuity of f or integrate over an infinite interval. Indefinite means that we do not specify bounds.

16) [T] [F] If $f(-5) = 0$ and $f(5) = 10$ then $f' = 1$ somewhere on the interval $[-5, 5]$.

Solution:
Yes this is the mean value theorem.

17) [T] [F] The sum $\frac{1}{n}\sum_{k=0}^{n-1} \frac{k}{n} = \frac{1}{n}[\frac{0}{n} + \frac{1}{n} + \cdots + \frac{n-1}{n}]$ is a Riemann sum to the integral $\int_0^1 x\ dx$.

Solution:
Yes, this is the Riemann sum.

18) [T] [F] The anti-derivative of $\mathrm{sinc}(x) = \sin(x)/x$ is equal to $\sin(\log(x)) + C$.

Solution:
Differentiate the right hand side to see that this is not true

19) [T] [F] The anti-derivative of $\log(x)$ is $1/x + C$.

20) [T] [F] We have $\int_0^x tf(t)\,dt = x\int_0^x f(t)\,dt$ for all functions f.

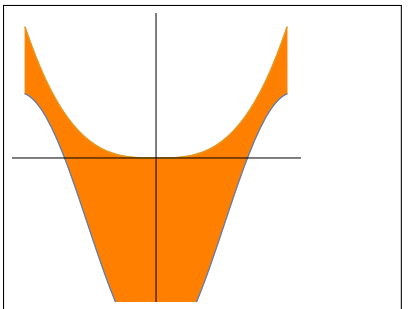
Solution:
It is already false for the constant function $f(t) = 1$.

Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the integrals with the pictures.

Integral	Enter 1-6
$\int_{-1}^{1}(1-x)^2\,dx$	
$\int_{-1}^{1}\lvert x\rvert\,dx$	
$\int_{-1}^{1}x^4\,dx$	

Integral	Enter 1-6
$\int_{-1}^{1}\lvert x\rvert^3-\cos(3x)\,dx$	
$\int_{-1}^{1}[\sin^2(\pi x)-\cos^2(\pi x)]\,dx$	
$\int_{-1}^{1}1-\lvert x\rvert\,dx$	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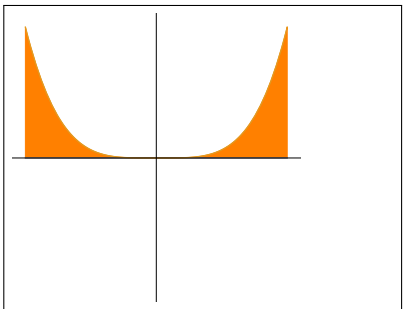
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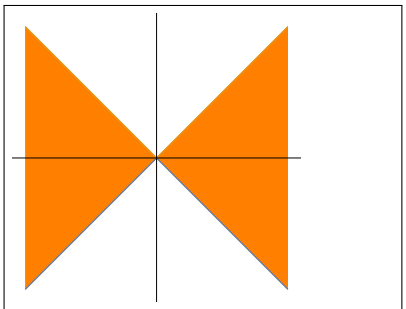
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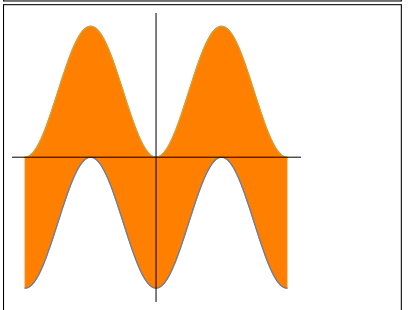
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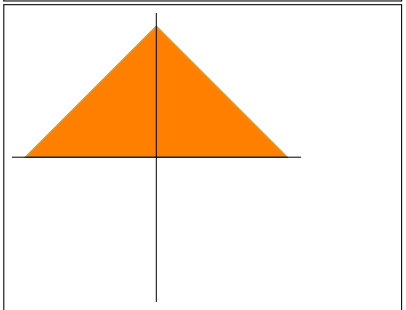
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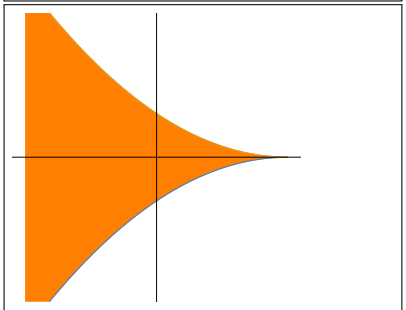
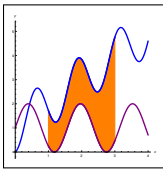
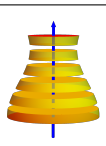

6)

Solution:
6,3,2,1,4,5.

b) (4 points) Match the concepts: each of the 4 figures illustrates one of the formulas which are the centers of the **mind map** we have drawn for this exam:


1)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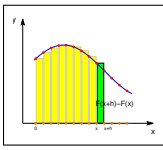
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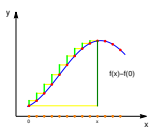

4)

Formula	Enter 1-4
$\int_a^b A(z)\,dz$	
$\int_a^b g(x)-f(x)\,dx$	

Formula	Enter 1-4
$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$	
$\int_0^x f'(t)\,dt = f(x)-f(0)$	

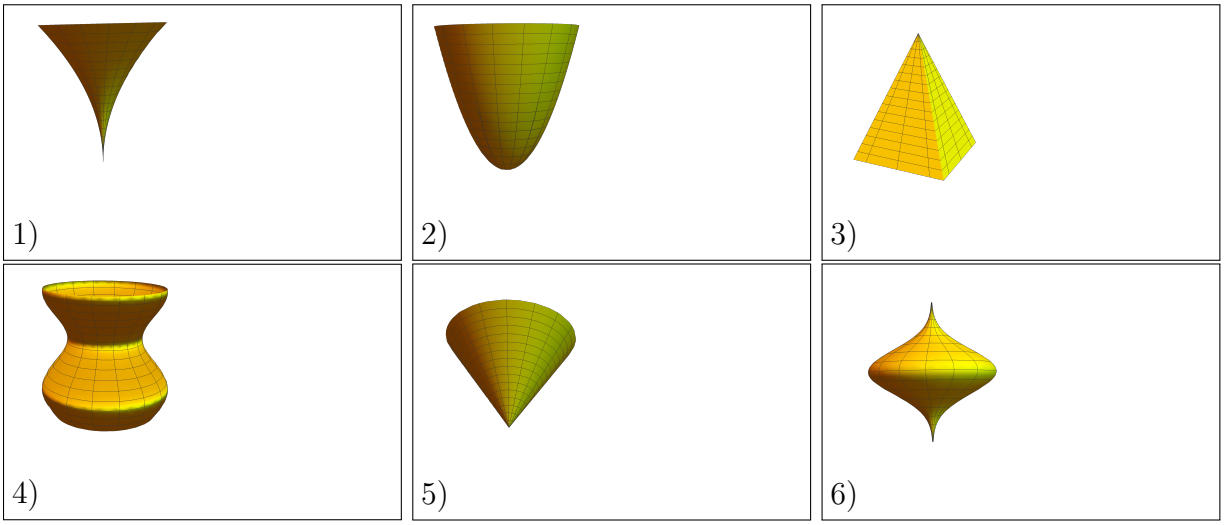
Solution:
2,1 and 3,4

Problem 3) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the volumes of solids.

Integral	Enter 1-6
$\int_0^1 \pi z^4\,dz$	
$\int_0^1 \pi z\,dz$	
$\int_0^1 \pi(4+\sin(4z))\,dz$	

Integral	Enter 1-6
$\int_{-1}^1 \pi e^{-4z^2}\,dz$	
$\int_0^1 \pi z^2\,dz$	
$\int_0^1 (1-z)^2\,dz$	



Solution:
1,2,4,6,5,3

b) (4 points) Fill in the missing word which links **applications** of integration.

The probability density function is the		of the cumulative distribution function.
The total cost is the		of the marginal cost.
The volume of a solid is the		of the cross section area function.
The velocity of a ball is the		of the acceleration of the ball.

Solution:
derivative, antiderivative, antiderivative, antiderivative

Problem 4) Area computation (10 points)

Find the area of the region enclosed the graphs of $y = x^4 - 12$ and $y = 8 - x^2$.

Solution:
The two curves intersect at $x = 2$ and $x = -2$.

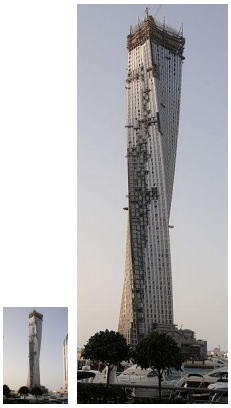
$$\int_{-2}^{2} 8 + x^2 - (x^4 - 12)\, dx = ... = \frac{928}{15}$$

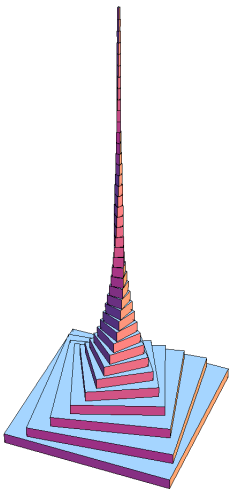
Problem 5) Volume computation (10 points)

The **infinity tower** in Dubai of height 330 meters has floors which can rotate. After much delay, it is expected to be completed this year. Inspired by the name "infinity", we build a new but twisted science center for which the side length of the square floor is

$$l(z) = \frac{1}{1+z} .$$

Find the volume of this new **Harvard needle building** which extends from 0 to ∞. We are the best!





Solution:
The cross section area is $A(z) = 1/(1+z)^2$. The integral is

$$\int_0^\infty \frac{1}{(1+z)^2}\, dz = -\frac{1}{1+z}|_0^\infty = 1$$

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. You should get a definite real number in each case.

a) (2 points) $\int_0^\infty xe^{-x}\, dx$

b) (3 points) $\int_0^1 5x^{1/5} + 3x^3 + 3\sqrt{1+x^2}x\, dx$.

c) (3 points) $\int_{-1}^1 \frac{1}{1+x^2}\, dx$

d) (2 points) $\int_0^{e-1} \frac{2}{1+x}\, dx$

Solution:
a) Use integration by parts 1
b) Use substitution in the third part $1 + 1 + \sqrt{8} - 1$
c) Use partial fractions $1/(2-x) + 1/(2+x)$ then integrate $\log(2+x) - \log(2-x)$. The answer is $\log(9)$. d) Direct integration 2

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int \frac{3}{\sqrt{1+3x}} + \cos^2(x)\, dx$
b) (3 points) $\int e^{x/5} - 7x^6 + \frac{4x}{x^2+1}\, dx$
c) (2 points) $\int \frac{4}{e^{4x+5}} + 3(x-1)\sin(x)\, dx$
d) (3 points) $\int \frac{1}{\sin^2(x)} + \frac{4}{x^2-1}\, dx$

Solution:
a) Use the double angle in the second part $2\sqrt{1+3x} + x/2 + \sin(2x)/4 + C$.
b) Use substitution in the third integral $5e^{x/5} - x^7 + 2\log(x^2+1) + C$.
c) Use parts in the second integral $-e^{-4x-5} + 3\sin(x) - 3(x-1)\cos(x) + C$.
d) Use partial fractions in the second part $-\cot(x) - 2\log(1+x) + 2\log(1-x) + C$.

Problem 8) PDF's and CDF's (10 points)

In order to verify that if f is a PDF and g is a PDF, then $(f+g)/2$ is a PDF, which three conditions do we have to check?

a) The function is piecewise

b) The function is .

c) The integral [ ] is equal to [ ].

Solution:

Problem 9) Catastrophes (10 points)

Verify first for each of the following functions that $x = 0$ is a critical point. Then give a criterium for stability of $x = 0$. The answer will depend on c.

a) (3 points) $f(x) = x^5 + 2x^2 - cx^2$.
b) (3 points) $f(x) = x^4 + cx^2 - x^2$.

Determine now in both examples for which parameter c the catastrophe occurs

c) (2 points) in the case $f(x) = x^5 + 2x^2 - cx^2$.
d) (2 points) in the case $f(x) = x^4 + cx^2 - x^2$.

Solution:
a) $f'(x) = 5x^4 + 4x - 2xc$ has $x = 0$ as a root. Its stability is determined by $f''(0) = 4 - 2c$.
b) $f'(x) = 4x^3 + 2cx - 2x$ has $x = 0$ as a root. Its stability is determined by $f''(0) = 2c - 2$.
c) For $c < 2$ the point $x = 0$ is a local minimum. For $c > 2$ it is a local maximum. $c = 2$ is a catastrophe.
d) For $c < 1$ the point $x = 0$ is a local minimum. For $c > 1$ it is a local maximum. $c = 1$ is a catastrophe.