Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

4/9/2021: Second hourly, Practice C

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F If f is a continuous function then $\int_0^x f(t)\,dt$ is an area and therefore positive.

2) T F The anti-derivative of $\operatorname{arccot}(x)$ is $-\log(\sin(x)) + C$.

3) T F The fundamental theorem of calculus implies that $\int_0^3 f''(x)\,dx = f'(3) - f(0)$.

4) T F The volume of a cylinder of height 3 and radius 5 is given by the integral $\int_0^3 \pi 5^2\,dx$.

5) T F The anti-derivative of $\tan(x)$ is $1/\cos^2(x)$.

6) T F The mean value theorem implies that the derivative of $\sin(x)$ in the interval $[0, \pi/2]$ is $2/\pi$ somewhere.

7) T F The function $F(x) = \int_0^x \sin(t^2)\,dt$ has the derivative $\sin(x^2)$.

8) T F If f is a PDF, then $\int_{-\infty}^{\infty} xf(x)\,dx$ is called the mean.

9) T F The identity $\frac{d}{dx}\int_0^1 \sin(x)\,dx = \sin(1)$ holds.

10) T F If a solid is scaled by a factor 2 in all directions then its volume increases by a factor 8.

11) T F If f is a PDF, then $\int_0^2 f(x)\,dx$ is the probability that the data are in $[0, 2]$.

12) T F If $f(x)$ is smaller than $g(x)$ for all x, then $\int_0^1 f(x) - g(x)\,dx$ is negative.

13) T F Every improper integral defines an infinite area.

14) T F The anti derivative of $f'(x)$ is equal to $f(x) + c$.

15) T F Catastrophes can explain why minima can change discontinuously.

16) T F If f is discontinuous at 0, then $\int_{-1}^1 f(x)\,dx$ is infinite.

17) T F If $f(-\infty) = 0$ and $f(\infty) = 1$ then $f' = 1$ somewhere on $(-\infty, \infty)$.

18) T F The anti-derivative of $1/x$ is $\log(x) + C$, where log is the natural log.

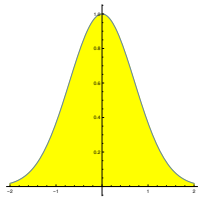
19) T F A catastrophe is defined as a critical point of f which is a minimum.

20) T F The integral $\int_0^\infty 1/x^2\,dx$ represents a finite area.

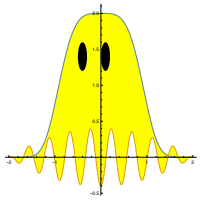
Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the following integrals with the regions. Graphs 1) and 2) are inspired by a cartoon by Matthew Freeman (J Epidemiol. Community Health. 2006 January; 60(1): 6)

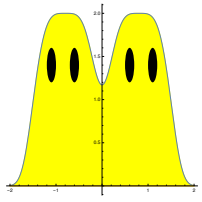
Integral	Fill in 1-4
$\int_{-2}^{2}(4-x^2)\cos^2(14x)/10-(4-x^2)\cos(14x)/15dx$	
$\int_{-2}^{2} 2\exp(-3(x+0.8)^4)+2\exp(-3(x-0.8)^4\ dx$	
$\int_{-2}^{2}\exp(-x^2)\ dx$	
$\int_{-2}^{2} 2\exp(-x^4)-(x^2-4)\cos(14x)/10\ dx$	



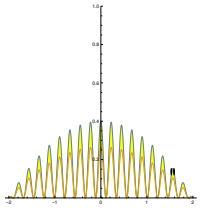
1) Normal distribution



2) Paranormal distribution



3) Abnormal distribution



4) Wormal distribution

b) (4 points) The intermediate value theorem assures that $f'(x)$ takes the value $(f(a)-f(b))/(b-a)$ somewhere in the interval (a,b) for $a<b$. Which of the following statements follows from this?

Result	Check
If $f(0)=-1$ and $f(1)=1$ then there is x with $0\leq x\leq 1$ with $f'(x)=2$	
If $f(0)=1$ and $f(1)=1$ then there is a critical point x of f in $(0,1)$	
If $f(0)=1$ and $f(1)=1$ then there is point where $f(x)=2$ in $(0,1)$	
If $f(0)=1$ and $f(1)=1$ then there is point where $f''(p)=0$ in $(0,1)$	

Problem 3) (10 points)

a) (4 points) Having seen some applications of integration and differentiation, complete the table: (the derivative of work is power, the anti derivative of marginal cost is cost.

Function f	Anti-derivative F
Probability density function	
	Total cost
	Mass
Area	
	Velocity
Power	
Velocity	

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Dissection method	
Newton method	

c) (2 points) Which is more general? Rolle's theorem assures that $f(a) = f(b)$, then $f'(x) = 0$ for some $a < x < b$. Related rates and implicit differentiation problems were not discussed this semester.

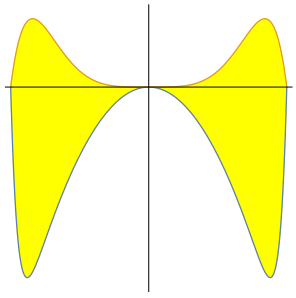
	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	

d) (2 points) Which integral is finite? Chose one!

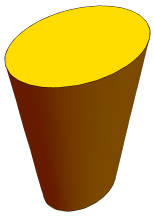
Integral	finite	infinite	
$\int_1^\infty 1/\sqrt{x}\, dx$			
$\int_1^\infty 1/x^2\, dx$			

Problem 4) Area computation (10 points)

The region enclosed by the graphs of $f(x) = x^{20} - x^2$ and $g(x) = x^4 - x^8$ is a cross section for a catamaran sailing boat. Find the area.

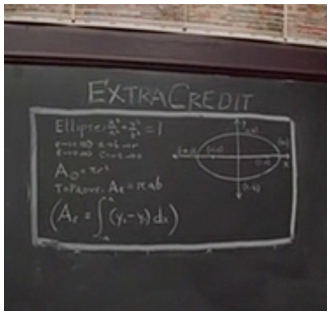


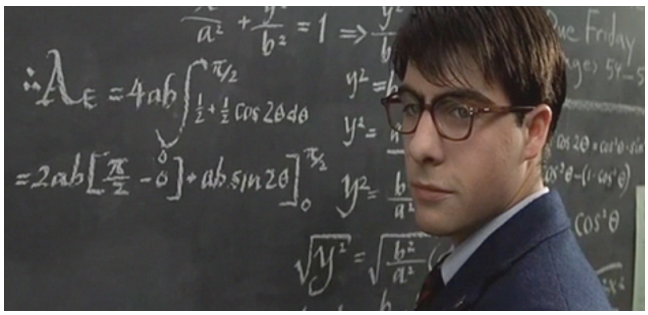
Problem 5) Volume computation (10 points)



We are given that an ellipse with diameters $2b$ and $2a$ has area πab. Find the volume of part of a cone whose height is between $z = 3$ and $z = 5$ for which the cross section at height z is an ellipse with parameters $a = 2z$ and $b = 3z$.

Remark. We will see later the area formula. In the movie **"Rushmore"**, the teacher tells about the problem: "I put that up as a joke. It's probably the hardest geometry equation in the world".





Screen shots from the movie Rushmore shows a blackboard where the formula for the ell.pdfe is computed using trig substitution. You might spot a double angle formula. We will come to that.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 (x-1)^4 \, dx$

b) (2 points) $\int_0^1 x \sin(\pi x) \, dx$.

c) (2 points) $\int_0^{\sqrt{3}} \frac{6}{1+x^2} \, dx + \int_0^{\sqrt{\pi}} \sin(x^2) 2x \, dx$

d) (2 points) $\int_1^2 \frac{1}{25-4x^2} \, dx$

e) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int e^{7x} - \log(x) \, dx$

b) (2 points) $\int \frac{5}{x+1} + 7\cos^2(x) \, dx$

c) (2 points) $\int \frac{11}{1+x^2} + 9\tan(x) \, dx$

d) (2 points) $\int \frac{4}{\cos^2(x)} + \frac{2}{\sin^2(x)} \, dx$

e) (2 points) $\int 2x \cos(x^2) \, dx$

Problem 8) PDF's and CDF's (10 points)

Verify that the function f which is defined to be $f(x) = 0$ for $x < 1$ and $f(x) = 3/x^4$ on $[1, \infty)$ is a PDF.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = x^6 - cx^4 - cx^2$, where c is a parameter.

a) (4 points) Verify that f has a critical point 0 for all c.

b) (3 points) Determine whether 0 is a minimum or maximum depending on c.

c) (3 points) For which c does a catastrophe occur?

Problem 10) Basic integrals (10 points)

Bonus problem: Find the anti derivatives. You have to solve in 10 seconds each. For every second over that limit, one point of the entire exam will be taken off. So, for example: if you use 62 seconds for the following 5 problems, you have used 12 seconds too much and 12 points are taken off from your exam. Don't worry, we do not assign negative points so that your final score will always remain a number between 0 and 110 points. But do not get too relaxed: if you choose not do this problem 10, 50 points are taken off.

a) (2 points) e^{-2x}.
b) (2 points) $\cos(15x)$.
c) (2 points) 2^x.
d) (2 points) $1/(1-x)$
e) (2 points) $1/(1+x^2)$