4/9/2021: Second hourly, Practice B

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.

- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

| 1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F — The function $f(x) = \int x \sin(x)\, dx$ is integrated using the method of partial fractions.

2) T F — For any continuous function f we have $\int_0^1 3f(t)\, dt = 3 \int_0^1 f(t)\, dt$.

3) T F — For any continuous function $\int_0^3 f(t)\, dt = 3 \int_0^1 f(t)\, dt$.

4) T F — For any continuous function $\int_0^1 1 - f(t)\, dt = 1 - (\int_0^1 f(t)\, dt)$.

5) T F — The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

6) T F — The fundamental theorem of calculus implies that $\int_1^3 f'(x)\, dx = f(3) - f(1)$.

7) T F — The integral $\pi \int_0^1 x^2\, dx$ gives the volume of a cone of height 1.

8) T F — The anti-derivative of $1/\cos^2(x)$ is $\tan(x)$.

9) T F — The function $F(x) = \int_0^x \tan(t^2)\, dt$ has the derivative $\tan(x^2)$.

10) T F — The function $f(x) = \sin(x)/2$ on $[0, \pi]$ and $f(x) = 0$ else is a PDF

11) T F — The identity $\frac{d}{dx} \int_1^2 \log(x)\, dx = \log(2) - \log(1)$ holds.

12) T F — If $f < 1$, then $\int_0^2 f(x)\, dx$ can be bigger than 1.

13) T F — An improper integral is an improperly defined definite indefinite integral.

14) T F — The anti derivative $F(x)$ of $f(x)$ satisfies $F'(x) = f(x)$.

15) T F — A parameter value c for which the number of minima are different for parameters smaller or larger than c is called a catastrophe.

16) T F — If f is unbounded at 0, then $\int_0^1 f(x)\, dx$ is infinite.

17) T F — If $f(-1) = 0$ and $f(1) = 1$ then $f' = 2$ somewhere on $(-1, 1)$.

18) T F — The anti-derivative of $\log(x)$ is $x \log(x) - x + C$, where log is the natural log.

19) T F — The sum $\frac{1}{n}[(\frac{0}{n})^2 + (\frac{1}{n})^2 + ... + (\frac{n-1}{n})^2]$ converges to $1/3$ in the limit $n \to \infty$.

20) T F — The **improper integral** $\int_1^\infty \frac{1}{x^2}\, dx$ represents a finite area.

Problem 2) Matching problem (10 points) No justifications are needed.

a) (4 points) Match the following integrals with the regions and indicate whether the integral represents a finite area.

Integral	Fill in 1-4	Finite?
$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\ dx$		
$\int_{-1}^{1} \frac{1}{x}\ dx$		
$\int_{-1}^{1} \frac{1}{1+x^2}\ dx$		
$\int_{-1}^{1} \log|x|\ dx$		

1)

2)

3)

4)
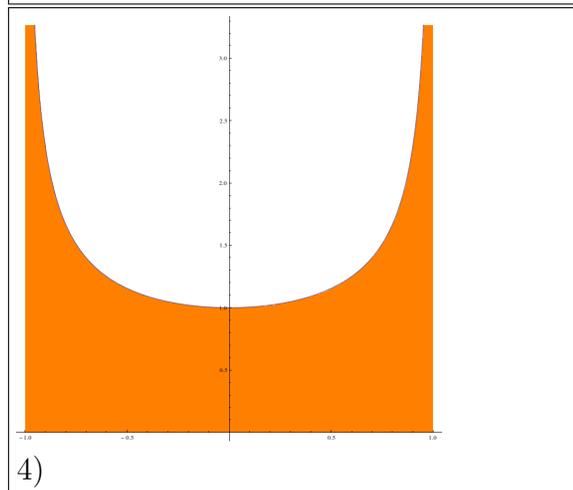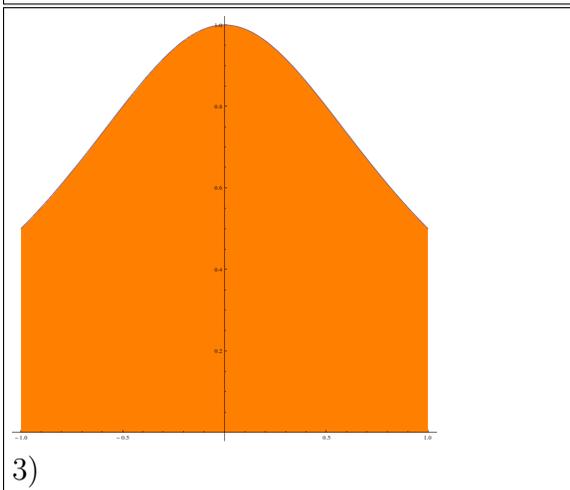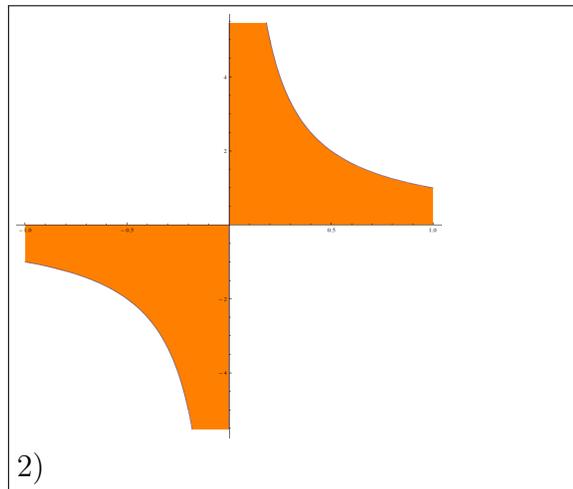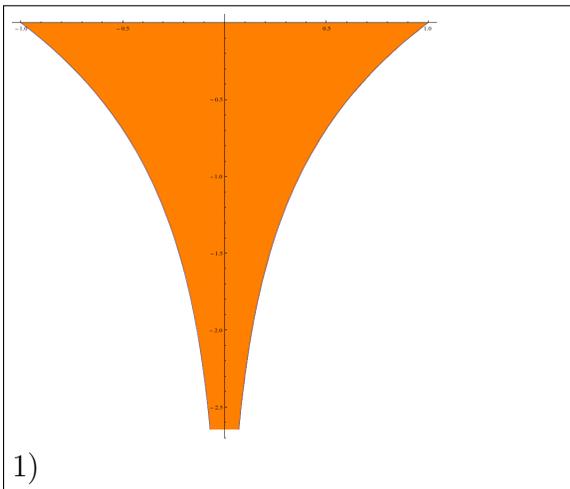
b) (6 points) Which of the following properties are always true. This means which are true for all choices of continuous functions and all choices of a, b, c.

Identity	Check if true
$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b f(x) + g(x)\,dx$	
$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$	
$\int_a^b f(x)^2\,dx = (\int_a^b f(x)\,dx)^2$	
$\int_a^a f(x)\,dx = 0$	
$\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$	

Problem 3) (10 points)

Fill in the missing part into the empty box to make a true statement:

a) (2 points)

$\frac{d}{dx}\int_0^x f(t)\,dt =$ ⬚ by the **fundamental theorem of calculus**.

b) (2 points)

$\int_0^x f(t)\,dt =$ ⬚ by the **fundamental theorem of calculus**.

c) (2 points)

The **method of** ⬚ writes $f(x) = 1/((x-6)(x+9))$ as $A/(x-6) + B/(x+9)$ and fixes the constants A, B.

d) (2 points)

A **probability distribution** satisfies $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and ⬚ for all x.

e) (2 points)

For an improper integral $\int_a^b f(x)\,dx$, either $a = \infty$ or $b = \infty$ or f is [   ] on $[a, b]$.
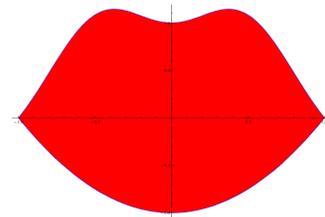

Problem 4) Area computation (10 points)

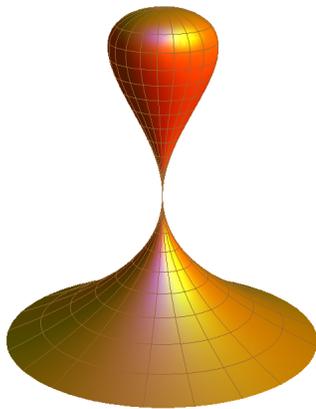The region enclosed by the graphs of
$$f(x) = x^2 - 1$$
and
$$g(x) = 1 - x^2 + (1 - \cos(2\pi x))/6$$
models of the lips of **Rihanna**. Find the area.

Problem 5) Volume computation (10 points)

The **kiss** is a solid of revolution for which the radius at height z is
$$z^2 \sqrt{1 - z}$$
and where $-1 \le z \le 1$. What is the volume of this solid? The name "kiss" is the official name for this quartic surface. Indeed, the top part has the shape of a **Hershey Kiss**. P.S. Creative **"exam product placement"** like this has been invented and patented by Oliver himself ...

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 \sqrt{1 + x}\,dx$.

b) (2 points) $\int_1^2 \frac{1}{9 - x^2}\,dx$

c) (2 points) $\int_2^e x \log(x)\, dx$

d) (2 points) $\int_1^2 \frac{x^2}{\sqrt{9-x^3}}\, dx$.

e) (2 points) $\int_0^\pi x^3 \sin(x)\, dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives:

a) (2 points) $\int 23 e^{23x} - x^{23}\, dx$

b) (2 points) $\int \frac{2}{x+3} + x^{1/23}\, dx$

c) (2 points) $\int \frac{23x}{1+x^2} + 23 \tan(x)\, dx$

d) (2 points) $\int \log(x)^2 dx$

e) (2 points) $\int \cos^2(3x)\, dx$

Jim Carrey in the movie "The number 23"

Problem 8) PDF's and CDF's (10 points)

Verify that the function $f(x) = \exp(-|x|)/2$ is a PDF.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = 2x^3 + cx^2$, where c is a parameter.

a) (2 points) Find the critical points of $f_3(x)$.

b) (2 points) Find the critical points of $f_{-3}(x)$.

c) (2 points) Check that 0 is always a critical point.

d) (2 points) For which c is 0 a minimum?

e) (2 points) For which c does the catastrophe occur?