# Unit 26: Partial fractions

LECTURE

The method of partial fractions is not really about integration. It is about algebra. We have learned how to integrate polynomials like $x^4 + 5x + 3$. What about rational functions? We will see here that they are a piece of cake if you know a bit about algebra.

**26.1.** Lets see what we know already:

- We also know that integrating $1/x$ gives $\log(x)$. We can for example integrate

$$\int \frac{1}{x-6} \, dx = \log(x - 6) + C \ .$$

- We also have learned how to integrate $1/(1+x^2)$. It was an important integral:

$$\int \frac{1}{1+x^2} \, dx = \arctan(x) + C \ .$$

Using substitution, we can do more like

$$\int \frac{dx}{1+4x^2} = \int \frac{du/2}{1+u^2} = \arctan(u)/2 = \arctan(2x)/2 \ .$$

- We also know how to integrate functions of the type $x/(x^2 + c)$ using substitution. We can write $u = x^2 + c$ and get $du = 2x dx$ so that

$$\int \frac{x}{x^2+c} \, dx = \int \frac{1}{2u} \, du = \frac{\log(x^2+c)}{2} \ .$$

- Also functions $1/(x+c)^2$ can be integrated using substitution. With $x + c = u$ we get $du = dx$ and

$$\int \frac{1}{(x+c)^2} \, dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{x+c} + C \ .$$

**26.2.** We would love to be able to integrate any rational function

$$f(x) = \frac{p(x)}{q(x)} \ ,$$

where $p, q$ are polynomials. This is where **partial fractions come in**. The idea is to write a rational function as a sum of fractions we know how to integrate. The above examples have shown that we can integrate $a/(x+c), (ax+b)/(x^2+c), a/(x+c)^2$ and cases, which after substitution are of this type.

> **Definition:** The **partial fraction method** writes $p(x)/q(x)$ as a sum of functions of the above type which we can integrate.

**26.3.** This is an algebra problem. Here is an important special case:

> In order to integrate $\int \frac{1}{(x-a)(x-b)} \, dx$, write
>
> $$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \ .$$
>
> and solve for $A, B$.

**26.4.** In order to solve for $A, B$, write the right hand side as one fraction again

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b) + B(x-a)}{(x-a)(x-b)} \ .$$

We only need to look at the nominator:

$$1 = Ax - Ab + Bx - Ba \ .$$

In order that this is true we must have $A + B = 0, Ab - Ba = 1$. This allows us to solve for $A, B$.

### Examples

**Example:** To integrate $\int \frac{2}{1-x^2} \, dx$ we can write

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

and integrate each term

$$\int \frac{2}{1-x^2} = \log(1+x) - \log(1-x) \ .$$

**Example:** Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} \ .$$

Now multiply out and solve for $A, B$:

$$A(x-2) + B(x-3) = 5 - 2x \ .$$

This gives the equations $A + B = -2$, $-2A - 3B = 5$. From the first equation we get $A = -B - 2$ and from the second equation we get $2B + 4 - 3B = 5$ so that $B = -1$ and so $A = -1$. We have not obtained

$$\frac{5 - 2x}{x^2 - 5x + 6} = -\frac{1}{x - 3} - \frac{1}{x - 2}$$

and can integrate:

$$\int \frac{5 - 2x}{x^2 - 5x + 6} \, dx = -\log(x - 3) - \log(x - 2) \ .$$

Actually, we could have got this one also with substitution. How?

**Example:** Integrate $f(x) = \int \frac{1}{1 - 4x^2} \, dx$. **Solution.** The denominator is factored as $(1 - 2x)(1 + 2x)$. Write

$$\frac{A}{1 - 2x} + \frac{B}{1 + 2x} = \frac{1}{1 - 4x^2} \ .$$

We get $A = 1/4$ and $B = -1/4$ and get the integral

$$\int f(x) \, dx = \frac{1}{4} \log(1 - 2x) - \frac{1}{4} \log(1 + 2x) + C \ .$$

**26.5.** There is a fast method to get the coefficients:

If $a$ is different from $b$, then the coefficients $A, B$ in

$$\frac{p(x)}{(x - a)(x - b)} = \frac{A}{x - a} + \frac{B}{x - b} \ ,$$

are

$$A = \lim_{x \to a}(x - a)f(x) = p(a)/(a - b), \ B = \lim_{x \to b}(x - b)f(x) = p(b)/(b - a) \ .$$

Proof. If we multiply the identity with $x - a$ we get

$$\frac{p(x)}{(x - b)} = A + \frac{B(x - a)}{x - b} \ .$$

Now we can take the limit $x \to a$ without peril and end up with $A = p(a)/(x - b)$.

**26.6.** Cool, isn't it? This **Hospital method** or **residue method** saves time especially with many functions where we would a complicated system of linear equations would have to be solved. I highly recommend you use this method.

Math is all about elegance. Avoid complicated methods if simple ones are available.

**26.7.** Here are examples:

**Example:** Find the anti-derivative of $f(x) = \frac{2x+3}{(x-4)(x+8)}$. **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2*4+3}{4+8} = 11/12$, and $B = \frac{2*(-8)+3}{(-8-4)} = 13/12$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8} .$$

The integral is

$$\frac{11}{12}\log(x-4) + \frac{13}{12}\log(x+8) .$$

**Example:** Find the anti-derivative of $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$. **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$ and $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$ and $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$. The integral is

$$\frac{3}{2}\log(x-1) - 7\log(x-2) + \frac{13}{2}\log(x-3) .$$

## HOMEWORK

As we had a wellness day break, this homework mixes problems from unit 25 and 26.

**Problem 26.1:** Do without the tabular method:
a) $\int xe^{7x+1} \, dx$.
b) $\int x^2 \log(x) \, dx$.

**Problem 26.2:** $\int (x-1)^7 \sin(3x) \, dx$. Do this with the tabular method.

**Problem 26.3:** a) $\int \cos(3x)\sin(5x) \, dx$ (Merry go round).
b) $\int \frac{1}{x^2-14x+45} \, dx$
c) $\int \frac{2}{x^2-9} \, dx$

**Problem 26.4:** $\int \frac{x^3-x+1}{x^2-1} \, dx$. Subtract first a polynomial.

**Problem 26.5:** $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)} \, dx$.