

INTRODUCTION TO CALCULUS

MATH 1A

Unit 24: Substitution

LECTURE

24.1. We know so far how to integrate functions like e^{6x} or $1/(1+x)$. The technique of **substitution** allows us to find more complicated anti-derivatives. If we differentiate the function $\sin(x^2)$ and use the chain rule, we get $\cos(x^2)2x$. The fundamental theorem of calculus tells us therefore that the anti-derivative of $\cos(x^2)2x$ is

$$\int \cos(x^2)2x \, dx = \sin(x^2) + C .$$

24.2. How can we see the integral without knowing the result already? Here is a very important case:

If $f(x) = g(u(x))u'(x)$, then the anti-derivative of f is $G(u(x)) + C$, where G is the anti-derivative of g .

Example: Find the anti derivative of

$$f(x) = e^{x^4+x^2}(4x^3 + 2x) .$$

Solution: It is $e^{x^4+x^2} + C$.

Example: Find

$$\int \sqrt{x^5+1}x^4 \, dx .$$

Solution. Try $(x^5+1)^{3/2}$ and differentiate. This gives $15/2$ of what we have. Therefore $F(x) = (2/15)(x^5+1)^{3/2}$.

Example: Find the anti-derivative of

$$\frac{\log(x)}{x} .$$

Solution: We spot that $1/x$ is the derivative of $\log(x)$. The anti-derivative is $\log(x)^2/2 + C$.

24.3. Writing down the function and adjusting a constant is the “speedy rule”:

If $\int f(ax + b) dx = F(ax + b)/a$ where F is the anti derivative of f .



Example: $\int \sqrt{x+1} dx$. **Solution:** $(x+1)^{3/2}(2/3) + C$.

Example: $\int \frac{1}{1+(5x+2)^2} dx$. **Solution:** $\arctan(5x+2)(1/5) + C$.

24.4. The method of substitution formalizes this: A) select part of the formula, call it u . B) then write $du = u'dx$. C) replace dx with du/u' . D) If all terms x have disappeared, integrate. E) Back substitute the variable x . If things should not work, go back to A) and try an other u .

$$\int f(u(x)) u'(x) dx = \int g(u) du .$$

24.5. We aim to end up with an integral $\int g(u) du$ which does not involve x anymore. Finally, after integration of this integral, do a **back-substitution**: replace the variable u again with the function $u(x)$.

Example: Find the anti-derivative of $\int \log(x)/x dx$. **Solution:** Pick $u = \log(x)$, $du = (1/x)dx$. Because $dx = xdu$, we get $\int u du = u^2/2 + C$. Back substitute to get $\log^2(x)/2 + C$.

Example: Find the anti-derivative

$$\int \log(\log(x)) \frac{1}{\log(x)x} dx .$$

Solution: Try $u = \log(x)$ and $du = (1/x)dx$, then plug this into the formula. It gives $\int \log(u)/u du$. We have just solved this integral before and got $\log(u)^2/2 + C$.

Example: Solve the integral

$$\int \frac{x}{1+x^4} dx .$$

Solution: Substitute $u = x^2$, $du = 2xdx$ gives $(1/2) \int du/(1+u^2) du = (1/2) \arctan(u) = (1/2) \arctan(x^2) + C$.

Example: What is the anti-derivative of $\sin(\sqrt{x})/\sqrt{x}$?

Solution. Try $u = \sqrt{x}$, $x = u^2$, $dx = 2u du$. The result is $-2 \cos(\sqrt{x}) + C$.

24.6. Here is an example that is more challenging

Example: Solve the integral

$$\int \frac{x^3}{\sqrt{x^2+1}} dx .$$

Solution. Trying $u = \sqrt{x^2+1}$ does not work. Try $u = x^2 + 1$, then $du = 2x dx$ and $dx = du/(2\sqrt{u-1})$. Substitute this in to get

$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}} du = \int \frac{(u-1)}{2\sqrt{u}} = \int u^{1/2}/2 - u^{-1/2}/2 du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2} .$$

24.7. When doing **definite integrals** $\int_a^b f(x) dx$, we could find the anti-derivative as described and then fill in the boundary points. Substituting the boundaries directly accelerates the process since we do not have to substitute back to the original variables:

$$\int_a^b g(u(x))u'(x) dx = \int_{u(a)}^{u(b)} g(u) du .$$

Proof. This identity follows from the fact that the right hand side is $G(u(b)) - G(u(a))$ by the fundamental theorem of calculus. The integrand on the left has the anti derivative $G(u(x))$. Again by the fundamental theorem of calculus the integral leads to $G(u(b)) - G(u(a))$.

Example: Find the anti-derivative of $\int_0^2 \sin(x^3 - 1)x^2 dx$. **Solution:**

$$\int_{x=0}^{x=2} \sin(x^3 + 1)x^2 dx .$$

Solution: Use $u = x^3 + 1$ and get $du = 3x^2 dx$. We get

$$\int_{u=1}^{u=9} \sin(u) du / 3 = (1/3) \cos(u)|_1^9 = [-\cos(9) + \cos(1)]/3 .$$

Example: $\int_0^1 \frac{1}{5x+1} dx = [\log(u)]/5|_1^6 = \log(6)/5$.

Example: $\int_3^5 \exp(4x - 10) dx = [\exp(10) - \exp(2)]/4$.

24.8. Substituting the bounds can sometimes be a bit tricky. An alternative way is to find first the anti derivative and then plug in the original bounds. Avoiding substituting the bounds actually is often the preferred way.

Example: $\int_3^5 \exp(4x - 10) dx = F(5) - f(3)$, where $F(x) = \exp(4x - 10)/4$.

Homework

Problem 24.1: Find the following anti-derivatives.

- a) $\int x^2 \sin(x^3) dx$
- b) $\int e^{x^6+x} (6x^5 + 1) dx$
- c) $-\cos(\sin(3x))/3$
- d) $e^{\tan(2x)} / \cos^2(2x)$.

Problem 24.2: Compute the following definite integrals. It is fine to find first the anti-derivative and only at the end place the bounds:

- a) $\int_1^2 \sqrt{x^5 + x} (5x^4 + 1) dx$
- b) $\int_0^{\sqrt{\pi}} 6 \sin(x^2) x dx$.
- c) $\int_e^{e^2} \frac{\sqrt{\log(x)}}{x} dx$.
- d) $\int_0^1 \frac{5x}{\sqrt{1+x^2}} dx$.

Problem 24.3: Find the definite integral

$$\int_e^{2e} \frac{dx}{\sqrt{\log(x)} x}.$$

Problem 24.4: a) Find the indefinite integral

$$\int \frac{x^5}{\sqrt{x^2 + 1}} dx.$$

b) Find the anti-derivative of

$$f(x) = \frac{1}{x(1 + \log(x)^2)}.$$

Problem 24.5: a) Find the anti-derivative of $\cos(x^3)/e^{\sin(x^3)} x^2$.

b) Find the anti-derivative of $\cot(\sqrt{x})/\sqrt{x}$.