INTRODUCTION TO CALCULUS

MATH 1A

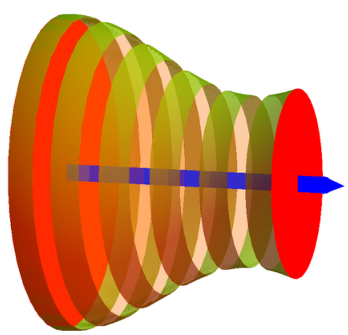
Unit 21: Volume

Lecture

21.1. To compute the **volume of a solid**, one can cut it into slices, so that each slice is perpendicular to a given line x. If $A(x)$ is the **area of the slice** and the body is enclosed between a and b then

$$V = \int_a^b A(x)\, dx$$

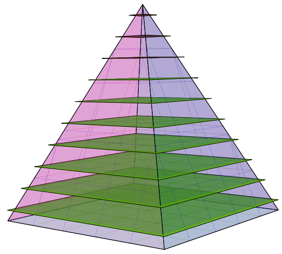
is the **volume** of the body. The integral adds up $A(x)dx$, the volume of the slices.



Example: Compute the volume of a pyramid with square base length 2 and height 2. **Solution**: we can assume the pyramid is built over the square $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. The cross section area at height h is $A(h) = (2-h)^2$. Therefore,

$$V = \int_0^2 (2-h)^2\, dh = \frac{8}{3}\ .$$

This is base area 4 times height 2 divided by 3.



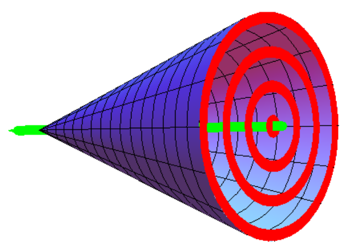
Definition: A **solid of revolution** is a surface obtained by rotating the graph of a function $f(x)$ around the x-axis.

The area of the cross section at x of a solid of revolution is $A(x) = \pi f(x)^2$. The volume of the solid is $\int_a^b \pi f(x)^2 \, dx$.

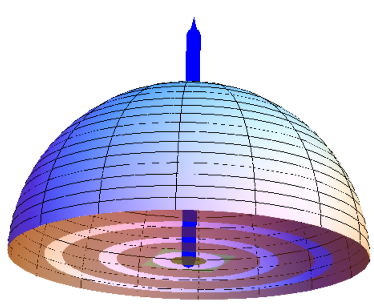
Example: Find the volume of a **round cone** of height 2 and where the circular base has the radius 1. **Solution**. This is a solid of revolution obtained by rotation the graph of $f(x) = x/2$ around the x axes. The area of a cross section is $\pi x^2/4$. Integrating this up from 0 to 2 gives

$$\int_0^2 \pi x^2/4 \, dx = \frac{x^3}{4 \cdot 3}|_0^2 = \frac{2\pi}{3} \ .$$

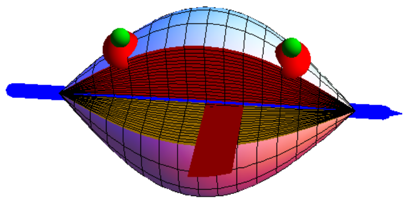
This is the height 2 times the base area π divided by 3.



Example: Find the volume of a **half sphere** of radius 1. **Solution:** The area of the cross section at height h is $\pi(1 - h^2)$.

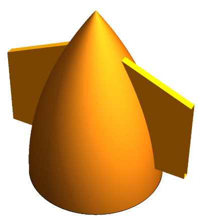


Example: If the function $f(x) = \sin(x)$ is rotated around the x axes, we get a **lemon**. But now we cut out a slice of $60 = \pi/3$ degrees as in the picture. Find the volume of the solid.
Solution: The area of a slice without the missing piece is $\pi \sin^2(x)$. The integral $\int_0^\pi \sin^2(x)\ dx$ is $\pi/2$ as derived in the lecture. Having cut out 1/6'th the area is $(5/6)\pi \sin^2(x)$. The volume is $\int_0^\pi (5/6)\pi \sin(x)^2\ dx = (5/6)\pi^2/2$.

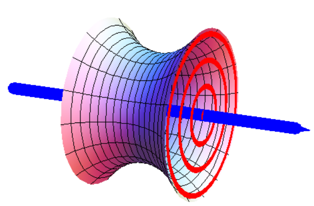


Homework

Problem 1: Space Ship SN10 just passed a high altitude test. We model the top of the rocket with a solid. Compute its volume for which the radius at position x is $9 - x^2$ and x ranges from 0 to 3.



Problem 2: A **catenoid** is the surface obtained by rotating the graph of $f(x) = \cosh(x) = (\exp(x) + \exp(-x))/2$ around the x-axes. We have seen that the graph of f is the chain curve, the shape of a hanging chain. Find the volume of of the solid enclosed by the catenoid between $x = -3$ and $x = 3$.
Hint. You might want to check first the identity $\cosh(x)^2 = (1 + \cosh(2x))/2$ using the definition $\cosh(x) = (\exp(x) + \exp(-x))/2$.



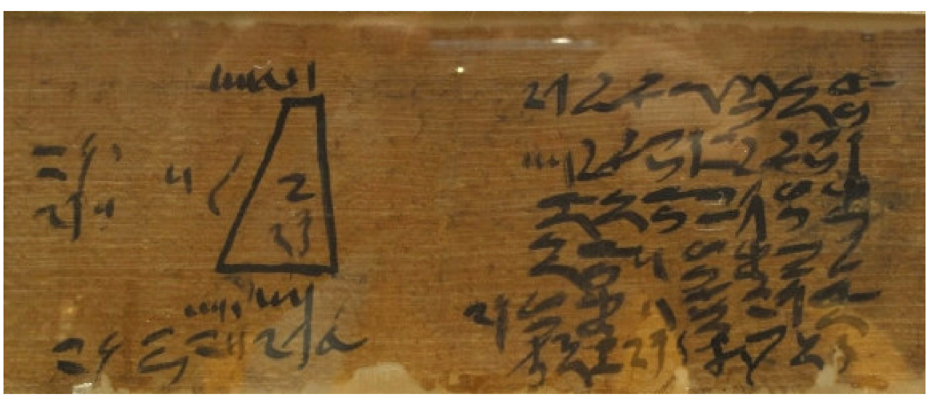
Problem 3: A **tomato** is given by $z^2 + x^2 + 4y^2 = 1$. If we slice perpendicular to the y axes, we get a circular slice $z^2 + x^2 \leq 1 - 4y^2$ of radius $\sqrt{1 - 4y^2}$. Find the area of this slice, then determine the volume of the tomato.

Problem 4: **Archimedes** was so proud of his formula for the volume of a sphere that he wanted the formula displayed on his tomb stone. To derive the formula, he wrote the volume of a half sphere of radius 1 as the difference between the volume of a cylinder of radius 1 and height 1 and the volume of a cone of base radius 1 and height 1. Relate the cross section area of the cylinder-cone complement with the cross section area of the sphere to recover his argument! No credit is given for screaming "Eureka".

Problem 5: Volumes were among the first quantities, Mathematicians wanted to measure and compute. One problem on **Moscow Egypt papyrus** dating back to 1850 BC explains the general formula $h(a^2 + ab + b^2)/3$ for a **truncated pyramid** with base length a, roof length b and height h. Verify that if you slice such a **frustrum** at height x, the area is $A(x) = (a + (b - a)x/h)^2$. Now use this to compute the volume using calculus.

Here is the translated formulation from the papyrus: [1] [2]

Remark: "You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4 result 16. You are to double 4 result 8. You are to square 2, result 4. You are to add the 16, the 8 and the 4, result 28. You are to take one-third of 6 result 2. You are to take 28 twice, result 56. See it is 56. You will find it right".



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[1] H. Eves, Great Moments in Mathematics, Vol. 1, MAA, Dolciani Math. Expos., 1980, p. 10
[2] Image Source: Carles Dorce, https://themathematicaltourist.wordpress.com