Unit 18: Fundamental theorem

Lecture

18.1. The **fundamental theorem of calculus** for differentiable functions will allow us to compute many integrals nicely. You have already made use of this theorem in the homework for today. Earlier in the course, we saw that $Sf(x) = h(f(0) + \cdots + f(kh))$ and $Df(x) = (f(x+h) - f(x))/h$ we have $SDf = f(x) - f(0)$ and $DSf(x) = f(x)$ if $x = nh$. This now becomes the **fundamental theorem**. It assumes that f' must be continuous.

$$\int_0^x f'(t)\,dt = f(x) - f(0) \text{ and } \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

Proof. Using notation of Euler, we write $A \sim B$. We say "A and B are close" and mean that $A - B \to 0$ for $h \to 0$. [1] From $DSf(x) = f(x)$ for $x = kh$ we have $DSf(x) \sim f(x)$ for $kh < x < (k+1)h$ because f is continuous. We also know $\int_0^x Df(t)\,dt \sim \int_0^x f'(t)\,dt$ because $Df(t) \sim f'(t)$ uniformly for all $0 \leq t \leq x$ by the definition of the derivative and the assumption that f' is continuous and using Bolzano on the bounded interval. We also know $SDf(x) = f(x) - f(0)$ for $x = kh$. By definition of the Riemann integral, $Sf(x) \sim \int_0^x f(t)\,dt$ and so $SDf(x) \sim \int_0^x Df(t)\,dt$.

$$f(x) - f(0) \sim SDf(x) \sim \int_0^x Df(t)\,dt \sim \int_0^x f'(t)\,dt$$

as well as

$$f(x) \sim DSf(x) \sim D\int_0^x f(t)\,dt \sim \frac{d}{dx}\int_0^x f(t)\,dt\,.$$

Example: $\int_0^5 x^7\,dx = \frac{x^8}{8}\Big|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

Example: $\int_0^{\pi/2} \cos(x)\,dx = \sin(x)|_0^{\pi/2} = 1$.

Example: Find $\int_0^{\pi} \sin(x)\,dx$. **Solution**: The answer is 2.

Example: For $\int_0^2 \cos(t+1)\,dt = \sin(x+1)|_0^2 = \sin(2) - \sin(1)$, the additional term $+1$ does not make matter as when using the chain rule, it goes away.

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[1] Bolzano or Weierstrass would write $A \sim B$ as $\forall \epsilon > 0, \exists \delta > 0, |h| < \delta \Rightarrow |A - B| < \epsilon$ but who can parse this?

Example: For $\int_{\pi/6}^{\pi/4} \cot(x)\ dx$, the anti-derivative is difficult to spot. It becomes only accessible if we know, where to look: the function $\log(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\log(\sin(x))|_{\pi/6}^{\pi/4} = \log(\sin(\pi/4)) - \log(\sin(\pi/6)) = \log(1/\sqrt{2}) - \log(1/2) = -\log(2)/2 + \log(2) = \log(2)/2$.
Let us look at two for now more challenging cases:
Example: The example $\int_2^3 2/(t^2-1)\ dt$ is challenging for now. We need a hint and write $2/(x^2-1) = 1/(x-1) - 1/(x+1)$. The function $F(x) = \log|x-1| - \log|x+1|$ has therefore $f(x) = 2/(x^2-1)$ as a derivative. The answer is $\int_2^3 2/(t^2-1)\ dt = F(3) - F(2) = \log(2) - \log(4) - \log(1) + \log(3) = \log(3) - \log(2) = \log(3/2)$.
Example: $\int_0^x \cos(\sin(x))\cos(x)\ dx = \sin(\sin(x))$ because the derivative of $\sin(\sin(x))$ is $\cos(\sin(x))\cos(x)$. The function $\sin(\sin(x))$ is an **anti-derivative** of f. If we differentiate this function, we get $\cos(\sin(x))\cos(x)$. Also this can be hard to spot for now. We will learn how to do this
We give reformulations of the fundamental theorem in ways in which it is mostly used: If f is the derivative of a function F then

$$\int_a^b f(x)\ dx = F(x)|_a^b = F(b) - F(a) .$$

In some textbooks, this is called the "second fundamental theorem" or the "evaluation part" of the fundamental theorem of calculus. The statement $\frac{d}{dx}\int_0^x f(t)\ dt = f(x)$ is the "anti-derivative part" of the fundamental theorem. They obviously belong together and are two different sides of the same coin.

Here is a version of the fundamental theorem, where the boundaries are functions of x. Given functions g, h and if F is a function such that $F' = f$, then

$$\int_{h(x)}^{g(x)} f(t)\ dt = F(g(x)) - F(h(x)) .$$

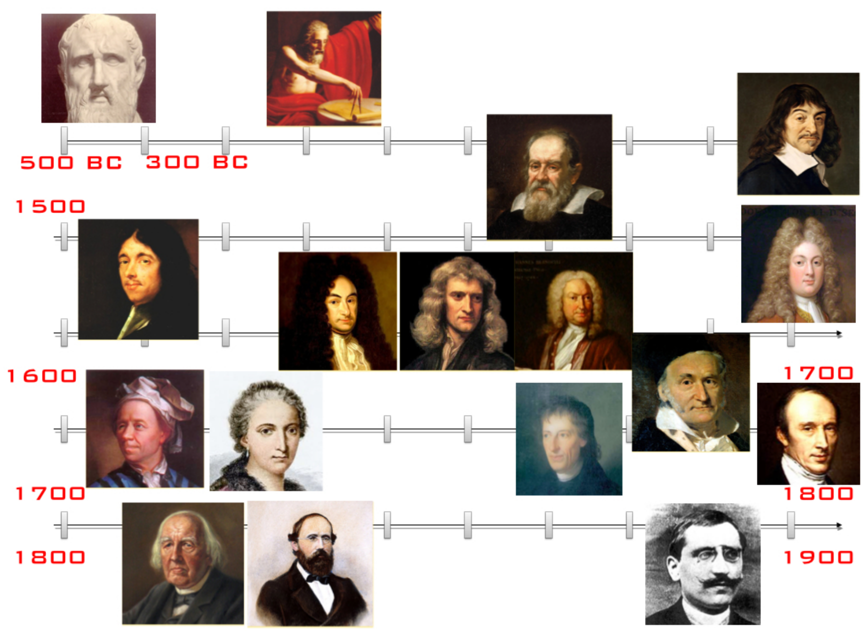
Example: $\int_{x^4}^{x^2} \cos(t)\ dt = \sin(x^2) - \sin(x^4)$.
The function F is called an **anti-derivative**. It is not unique but the above formula does always give the right result. Lets make a list You should have as many **anti-derivatives** "hard wired" in your brain. It really helps. Here are the core functions you should know.

function	anti derivative
x^n	$\frac{x^{n+1}}{n+1}$
$\sqrt{x}$	$\frac{x^{3/2}2}{3}$
e^{ax}	$\frac{e^{ax}}{a}$
$\cos(ax)$	$\frac{\sin(ax)}{a}$
$\sin(ax)$	$\frac{-\cos(ax)}{a}$
$\frac{1}{x}$	$\log(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\log(x)$	$x\log(x) - x$

Make your own table!

Newton ("Sir Slightly Annoyed") and Leibniz ("Mr Sour Face")

Isaac Newton and **Gottfried Leibniz** are the main figures for the fundamental theorem of calculus. You can see from the expression of their faces, they are not pleased that Oliver has added other calculus pioneers. The sour faces might also have to do with the fact that they have to live already for 10 years on the same handout with **Austin Powers** and **Doctor Evil**. Since 2021, they got reanimated with the Deep Heritage tool, which is an internet sensation these days.



Homework

Problem 18.1: Find a function F such that $F' = f$, then integrate
a) $\int_- 1^1 4x^{33} + 300x^2 \, dx$.
b) $\int_0^1 (x+1)^5 \, dx$.

Problem 18.2: Find a function F such that $F' = f$, then integrate:
a) $\int_2^3 5/(x-1) \, dx$, b) $\int_0^{\sqrt{\pi}} \sin(x^2) 4x \, dx$

Problem 18.3: Evaluate the following integrals:
a) $\int_1^2 2^x \, dx$
b) $\int_0^{\sqrt{3}} \frac{1}{1+x^2} \, dx$,

Problem 18.4: a) Compute $F(x) = \int_0^{x^3} \sin(t) \, dt$, then find $F'(x)$.
b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t) \, dt$ then find $G'(x)$

Problem 18.5: a) **A clever integral:** Evaluate the following integral (just by being clever, there is no algebra, and no work is needed):

$$\int_{-\pi}^{\pi} \sin(\sin(\sin(\sin(\sin(x))))) \, dx \, .$$

b) **An evil integral**: Evaluate $\int_e^{e^e} \frac{1}{\log(x)x} \, dx$.
Hint: Figure out a function $F(x)$ which satisfies $F'(x) = 1/(\log(x)x)$. Don't hesitate to ask MiniMe (Oliver).

Clever and evil

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