Unit 17: Riemann Integral

Lecture

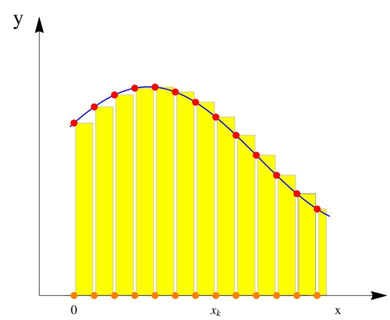
17.1. In this lecture, we define the definite integral $\int_0^x f(t)\, dt$ if f is a differentiable function. We then compute it for some basic functions. We have previously defined the **Riemann sums**

$$Sf(x) = h[\, f(0) + f(h) + f(2h) + \cdots + f(kh)] \;,$$

where k is the largest integer such that $kh < x$. Lets write S_n if we want to stress that the parameter $h = 1/n$ was used in the sum. We define the **Riemann integral** as the limit of these sums $S_n f$, when the **mesh size** $h = 1/n$ goes to zero.

Definition: Define

$$\int_0^x f(t)\, dt = \lim_{n \to 0} S_n f(x) \;.$$



17.2. A very important result is that

For any continuous function, the limit exists.

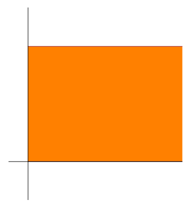
It is easier to see when f is f is differentiable as one can then estimate the error. There are n little pieces which are each of area $\leq M/n$, where M is the maximal slope that f can have in the given interval.

For non-negative f, the value $\int_0^x f(x)\, dx$ is the **area between the x-axis and the graph** of f. For general f, it is a **signed area**, the difference between two areas.

17.3. The Riemann integral is the limit $h \sum_{x_k = kh \in [0,x)} f(x_k)$. It converges to the area under the curve for all **continuous** functions. In probability theory, one uses also an other integral, the **Lebesgue integral**. It can be defined as the limit $\frac{1}{n} \sum_{k=1}^{n} f(x_k)$ where x_k are **random points** in $[0, x]$. This is a **Monte-Carlo integral** definition of the Lebesgue integral.

17.4. Riemann also looked also at points $x_0 < x_1 < \cdots < x_n$ $[0, x]$ such that the maximal distance $(x_{k+1} - x_k)$ between neighboring x_j goes to zero. The Riemann sum is then $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$, where y_k is arbitrarily chosen inside the interval (x_k, x_{k+1}). For continuous functions, the limiting result is the same the $Sf(x)$ sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated.

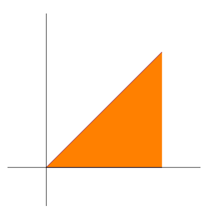
Example: If $f(x) = c$ is constant, then $\int_0^x f(t)\, dt = cx$. We can see also that $cnx/n \leq S_n f(x) \leq c(n+1)x/n$.



Example: Let $f(x) = cx$. The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$. Adding up the Riemann sum is more difficult. Let k be the largest integer smaller than $xn = x/h$. Then

$$S_n f(x) = \frac{1}{n} \sum_{j=1}^{k} \frac{cj}{n} = \frac{ck(k+1)/2}{n^2} .$$

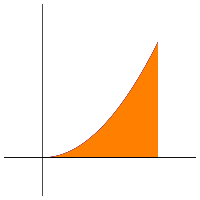
Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\, dt = cx^2/2$.

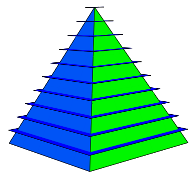


Example: Let $f(x) = x^2$. In this case,we can not see the numerical value of the area geometrically. But since we have computed $S[x^2]$ in the first lecture of this course and seen that it is $[x^3]/3$ and since we have defined $S_h f(x) \to \int_0^x f(t)\, dt$ for $h \to 0$ and $[x^k] \to x^k$ for $h \to 0$, we know that

$$\int_0^x t^2\, dt = \frac{x^3}{3} .$$

This example actually computes the **volume of a pyramid** which has at distance t from the top an area t^2 cross section. Think about $t^2 dt$ as a slice of the pyramid of area t^2 and height dt. Adding up the volumes of all these slices gives the volume.

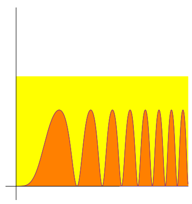




Linearity of the integral (see homework) $\int_0^x f(t)+g(t)\ dt = \int_0^t f(t)\ dt + \int_0^x g(t)\ dt$ and $\int_0^x \lambda f(t)\ dt = \lambda \int_0^x f(t)\ dt$.

Upper bound: If $0 \leq f(x) \leq M$ for all x, then $\int_0^x f(t)\ dt \leq Mx$.

Example: $\int_0^x \sin^2(\sin(\sin(t)))/x\ dt \leq x$. **Solution**. The function $f(t)$ inside the interval is non-negative and smaller or equal to 1 The graph of f is therefore contained in a rectangle of width x and height 1.



17.5. We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If f and g satisfy $|f(x) - g(x)| \leq c$, then

$$\int_0^x |f(x) - g(x)|\ dx \leq cx\ .$$

17.6. We know identities like $S_n[x]_h^n = \frac{[x]_h^{n+1}}{n+1}$ and $S_n \exp_h(x) = \exp_h(x)$ already. Since $[x]_h^k - [x]^k \to 0$ we have $S_n[x]_h^k - S_n[x]^k \to 0$ and from $S_n[x]_h^k = [x]_h^{k+1}/(k+1)$.
The other equalities are the same since $\exp_h(x) = \exp(x) \to 0$. This gives us:

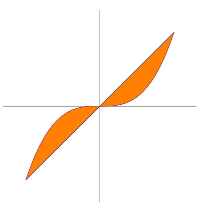
$\int_0^x t^n\ dt = \frac{x^{n+1}}{n+1}$ $\int_0^x \cos(t)\ dt = \sin(x)$
$\int_0^x e^t\ dt = e^x - 1$ $\int_0^x \sin(t)\ dt = 1 - \cos(x)$

Homework

In the following homework you can use that $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$ if F is a function which satisfies $F'(x) = f(x)$. We have already verified the identity for sums.

Problem 17.1: a) What is the integral $\int_0^1 4x^{24}\,dx$?
b) Find the integral $\int_0^{\log(2)} 5e^t\,dt$.
c) Calculate $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$.
d) Find $\int_0^{\pi/2} \cos^2(t)\,dt$.
e) Find $\int_0^{\pi/2} \sin^4(t)\,dt$.

Problem 17.2: The region enclosed by the graph of x and the graph of x^5 has a propeller type shape. Find its (positive) area.



Problem 17.3: Make a geometric picture for each of the following statements (which are rules for integration):

- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
- $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$.
- $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$.

Problem 17.4: Here are some more challenging integrals. Maybe you have to guess or remember some old computations, maybe even look at the exam. a) $\int_0^2 x^x(1+\log(x))\,dx$
b) $\int_0^1 (3/2)\sqrt{1+x}\,dx$
c) $\int_0^{\sqrt{\log(2)}} 4xe^{-x^2}\,dx$
d) $\int_1^e 5\log(x)/x\,dx$

Problem 17.5: In this problem, it is crucial that you plot the function first. Split the integral up into parts. Find $\int_{-1}^4 f(x)\,dx$ for $f(x) = |x - |x-2||$.