**INTRODUCTION TO CALCULUS**

MATH 1A

# Unit 12: Global extrema

## Lecture

**12.1.** In this lecture we are interested in global maxima. These are points where the function is maximal overall. These **global extrema** can occur either at critical points of $f$ or at the boundary of the domain, where both $f$ and $f'$ are defined.

> **Definition:** A point $a$ is called a **global maximum** of $f$ if $f(a) \geq f(x)$ for all $x$. It is called a **global minimum** of $f$ if $f(a) \leq f(x)$ for all $x$.

**12.2.** How do we find global maxima? The answer is simple: make a list of all local extrema and boundary points, then pick the largest. Global maxima or minima do not need to exist however. The function $f(x) = x^2$ has a global minimum at $x = 0$ but no global maximum. The function $f(x) = x^3$ has no global maximum nor minimum at all. We can however look at global maxima on finite intervals.

**Example:** Let us look at the example from last week where we found the square of maximal area among all squares of side length $x, 1-x$. The function $f(x) = x(1-x)$ had a maximum at $x = 1/2$. We also have to look at the boundary points. Why? Because both $x$ and $f(x)$ can not become negative. We see that $f(x)$ has to be looked at on the interval $[0,1]$. We write $[0,1]$ if we mean that 0 and 1 are included. To decide about global maxima, just look at the critical points and boundary points and pick the maximal.

**Example:** Find the global maximum of $f(x) = x^2$ on the interval $[-1,2]$. **Solution**. We look for local extrema at critical points and at the boundary. Then we compare all these extrema to find the maximum or minimum. The critical points are $x = 0$. The boundary points are $-1, 2$. Comparing the values $f(-1) = 1, f(0) = 0$ and $f(4) = 4$ shows that $f$ has a global maximum at 2 and a global minimum at 0.

> **Extreme value theorem of Bolzano**: A continuous function $f$ on a closed finite interval $[a,b]$ attains a global maximum and a global minimum.

**Proof**: for every $n$, make a list of the points $x_k = (a+(k/n)(b-a))$ where $k = 1, \dots, n$. Pick the one where $f(x_k)$ is maximal one and call this $x_n$. Now we use the **Bolzano-Weierstrass theorem** which assures that any sequence of numbers $x_n$ on a closed interval $[a,b]$ has an accumulation point. Such an accumulation point is a maximum. Similarly, we can construct the minimum.

The **Bolzano-Weierstrass theorem** is verified constructively too: cut the interval in two equal parts and choose a part which contains infinitely many points $x_n$. We have reduced the problem to a smaller interval. Now take this interval and again divide it into two. Relabel the points there with $x_n$. Again chose the one in which $x_n$ hits infinitely many times. Washing, rinse and repeating this again and again leads to smaller and smaller intervals of size $[b-a]/2^n$ in which there are infinitely many points. Note that these intervals are nested so that they lead to a limit (if the interval were $[0,1]$ and we would cut each time into 10 pieces, then we would gain in every step one digit of the decimal expansion of the number we are looking for).

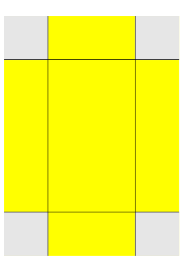
**12.3.** Note that the global maximum or minimum can also be on the boundary or points where the derivative does not exist:

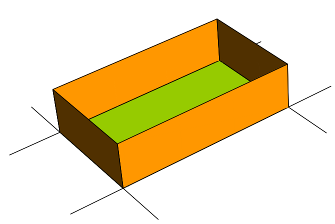
**Example:** Find the global maximum and minimum of the function $f(x) = |x|$. The function has no absolute maximum as it goes to infinity for $x \to \infty$. The function has a global minimum at $x = 0$ but the function is not differentiable there. The point $x = 0$ is a point which does not belong to the domain of $f'$.

**Example:** A **soda can** is a cylinder of volume $\pi r^2 h$. The surface area $2\pi rh + 2\pi r^2$ measures the amount of material used to manufacture the can. Assume the surface area is $2\pi$, we can solve the equation for $h = (1-r^2)/r = 1/r - r$ **Solution:** The volume is $f(r) = \pi(r - r^3)$. Find the can with maximal volume: $f'(r) = \pi - 3r^2\pi = 0$ showing $r = 1/\sqrt{3}$. This leads to $h = 2/\sqrt{3}$.

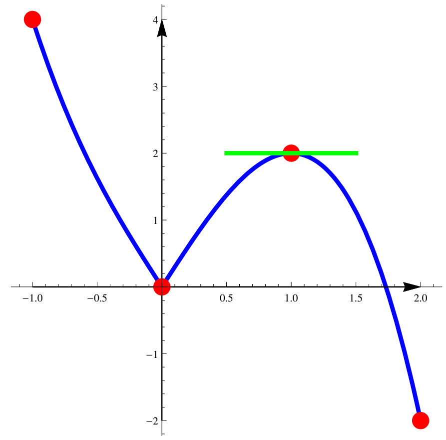
**Example:** Take a card of $2 \times 2$ inches. If we cut out 4 squares of equal side length $x$ at the corners, we can fold up the paper to a tray with width $(2-2x)$ length $(2-2x)$ and height $x$. For which $x \in [0,1]$ is the tray volume maximal?

**Solution** The volume is $f(x) = (2-2x)(2-2x)x$. To find the maximum, we need to compare the critical points which is at $x = 1/3$ and the boundary points $x = 0$ and $x = 1$.





**Example:** Find the global maxima and minima of the function $f(x) = 3|x| - x^3$ on the interval $[-1, 2]$.
**Solution.** For $x > 0$ the function is $3x - x^3$ which can be differentiated. The derivative $3 - 3x^2$ is zero at $x = 1$. For $x < 0$ the function is $-3x - x^3$. The derivative is $-3 - x^2$ and has no root. The only critical points are 1. There is also the point $x = 0$ which is not in the domain where we can differentiate the function. We have to deal with this point separately. We also have to look at the boundary points $x = -1$ and $x = 2$. Making a list of function values at $x = -1, x = 0, x = 1, x = 2$ gives the maximum.



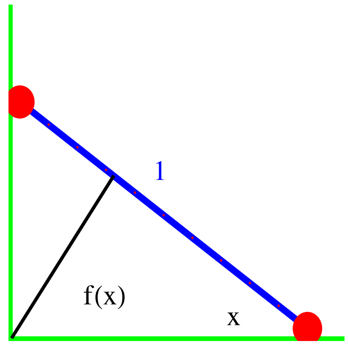

## Homework

**Problem 1:** Find all the local maxima and minima as well as the global maximum and the global minimum of the function $f(x) = x^4 - 4x^3 - 2x^2 + 12x$ on the closed interval $[-4, 4]$. Make sure to compute the critical points inside the interval and then compare also the boundary points.

**Problem 2:** Find the global maximum and minimum of the function $f(x) = 2\sin(x) + x$ on the interval $[-10, 10]$.

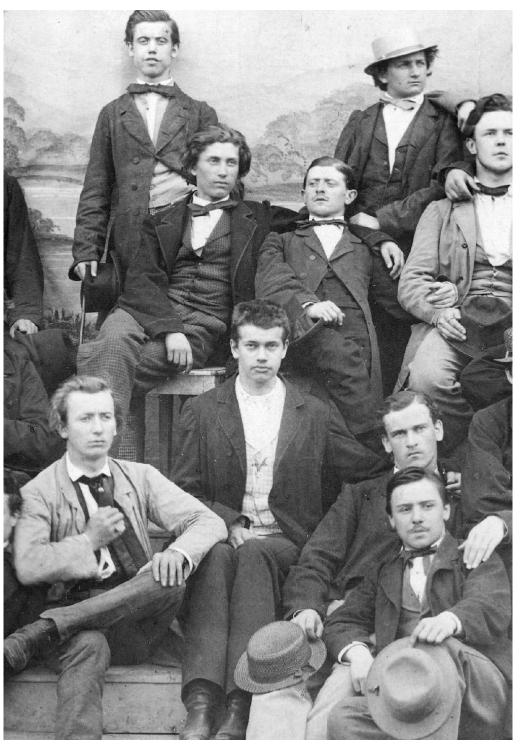
**Problem 3:** Mathcandy.com (look it up!) manufactures spherical candies. Its effectiveness is $A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius $r$. Find the radius, where $f(r) = A(r) - V(r)$ has a global maximum for $r \geq 0$.

**Problem 4:** A ladder of length 1 is one side at a wall and on one side at the floor. a) Verify that the distance from the ladder to the corner is $f(x) = \sin(x)\cos(x)$.
b) Find the angle $x$ for which $f(x)$ is maximal.




**Problem 5:** a) The function $S(x) = -x\log(x)$ is called the **entropy function**. Find the probability $0 < x \leq 1$ which maximizes entropy.
b) Find the global minimum of the **Helmholtz free energy** $G = H - TS$, where $T = 10$ is temperature, $S(x)$ is the entropy function in a) and $H = x$ is the **internal energy**.
P.S. One of the most important principles in science is that nature tries to maximize entropy or minimize fee energy.

Entropy has been introduced by Ludwig Boltzmann. It is important in physics and chemistry. $S = k\log(W)$ is interpreted as $W = 1/p$, then take the expectation giving $S = -k\sum_p p\log(p)$. Note the use of log and not ln. Hermann von Helmholz (1812-1894).



Boltzmann (1844-1906)

Helmholtz (1812-1894).