INTRODUCTION TO CALCULUS

MATH 1A

Unit 8: Derivative Function

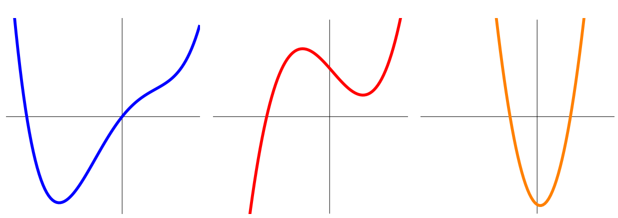
LECTURE

8.1. The derivative $f'(x) = \frac{d}{dx} f(x)$ was defined as a limit of $(f(x+h) - f(x))/h$ for $h \to 0$. We have seen that $\frac{d}{dx} x^n = nx^{n-1}$ holds for integer n. We also know already that $\sin'(ax) = a\cos(x), \cos'(ax) = -a\sin(x)$ and $\exp'(ax) = a\exp(ax)$. We can now differentiate already a lot of functions and evaluate the derivative $f'(x)$ at a given point x and compute the slope of the graph of f at x.

8.2. Example: Find the derivative $f'(x)$ of $f(x) = \sin(4x) + \cos(5x) - \sqrt{x} + 1/x + x^4$ and evaluate it at $x = 1$. **Solution**: $f'(x) = 4\cos(4x) - 5\sin(5x) - 1/(2\sqrt{x}) - 1/x^2 + 4x^3$. Plugging in $x = 1$ gives $-\pi - 1/2 - 1 + 4$.

8.3. The differentiation process produces also a rule which assigns to a function f a new function f', the **derivative function**. For example, for $f(x) = \sin(x)$, we get $f'(x) = \cos(x)$. In this lecture, we want to understand the new function and its relation with f. What does it mean if $f'(x) > 0$? What does it mean that $f'(x) < 0$? Do the roots of f tell about f' or do the roots of f' tell about f?

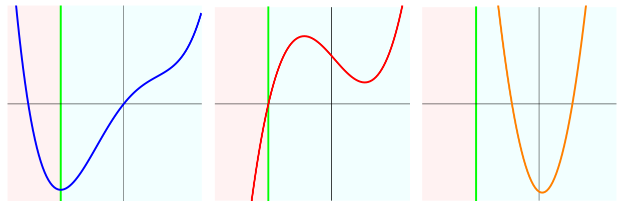
8.4. Here is an example of a function f, its derivative f' and the derivative of the derivative f''. Can you see the relation?



8.5. To understand this, it is good to distinguish intervals, where $f(x)$ is **increasing** or **decreasing**. This are the intervals where $f'(x)$ is positive or negative.

> **Definition:** A function is called **strictly monotonically increasing** on an interval $I = (a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$. It is **strictly monotonically decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Monotonically increasing functions "go up" when you "increase x". Lets color that:



Example: Can you find a function f on the interval $[0,1]$ which is bounded $|f(x)| \leq 1$ but such that $f'(x)$ is unbounded?

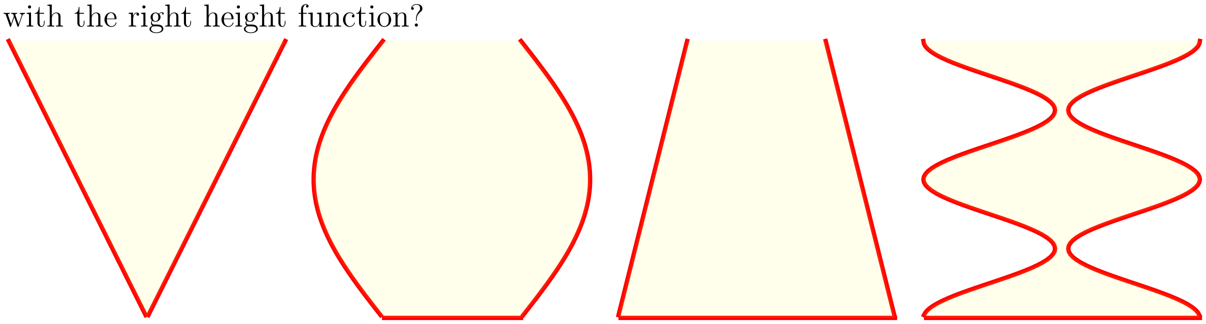
Definition: Given $f(x)$, we can define $g(x) = f'(x)$ and then take the derivative g' of g. This second derivative $f''(x)$ is called the **acceleration**. It measures the rate of change of the tangent slope. For $f(x) = x^4$, for example we have $f''(x) = 12x^2$. If $f''(x) > 0$ on some interval the function is called **concave up**, if $f''(x) < 0$, it is **concave down**.

Example: Find a function f which has the property that its acceleration is constant equal to 6. **Solution.** We have to get a function such that its derivative is $6x$. That works for $3x^2$.

Example: Find a function f which has the property that its acceleration f'' is equal to the negative of f. To do so, try some of the basic functions you know and compute $f'(x), f''(x)$ in each case.

8.6. In the famous **bottle calibration problem**, we fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time t. Assume the radius of the bottle is $f(z)$ at height z. Can you find a formula for the height $g(t)$ of the water? This is not so easy. But we can find the rate of change $g'(t)$. Assume for example that f is constant, then the rate of change is constant and the height of the water increases linearly like $g(t) = t$. If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of g and f. Before we look at this more closely, lets try to match the following cases of bottles with the graphs of the functions g qualitatively.

Example: In each of the bottles, we call g the height of the water level at time t, when filling the bottle with a constant stream of water. Can you match each bottle with the right height function?



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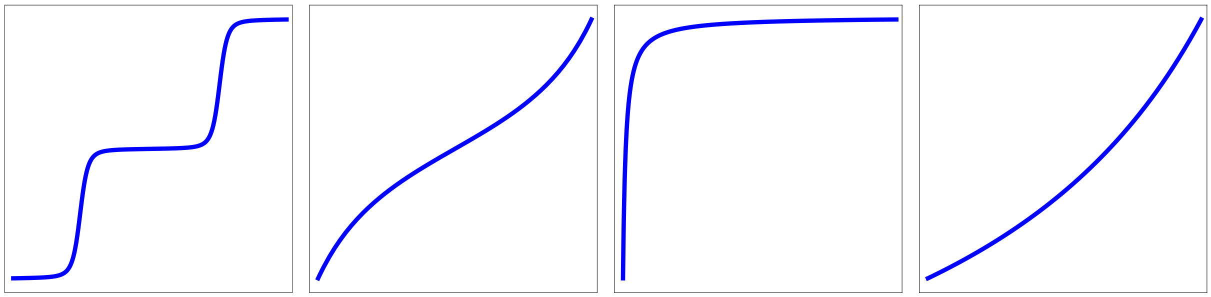


1) 2) 3) 4)

8.7. The key is to look at $g'(t)$, the rate of change of the height function. Because $[g(t+h) - g(t)]$ times the area πf^2 is a constant times the time difference $h = dt$, we have **bottle calibration formula**

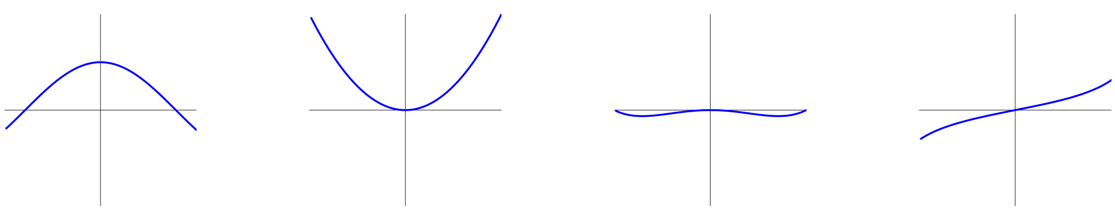
$$g' = \frac{1}{\pi f^2} \; .$$

It relates the derivative function of g with the thickness $f(t)$ of the bottle at height g. No need to learn this. It just explains the story completely. It tells that that if the bottle radius f is large, then the water level increase g' is small and if the bottle radius f is small, then the liquid level change g' is large.

Homework

Problem 8.1: a) Determine on which interval the function $x^4 - x^2$ is monotonically increasing or monotonically decreasing.
b) We write $f'(x)$ for the derivative, $f''(x) = f^{(2)}$ for the second derivative, $f'''(x) = f^{(3)}(x)$ for the third derivative etc. What is the 1000'th derivative of $\sin(x)$?

Problem 8.2: Match the following functions with their derivatives. Explain using monotonicity



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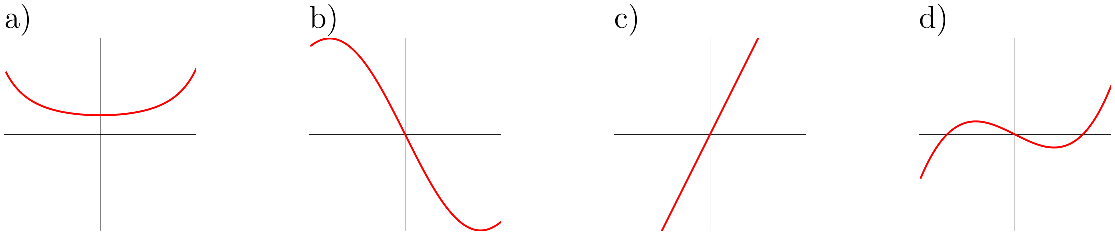

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Problem 8.3: Draw for the following functions the graph of the function $f(x)$ as well as the graph of its derivative $f'(x)$. You do not have to compute the derivative analytically as a formula here since we do not have all tools yet to compute the derivatives. The derivative function you draw needs to have the right qualitative shape however.

a) The **"To whom the bell tolls"** function

$$f(x) = e^{-x^2}$$

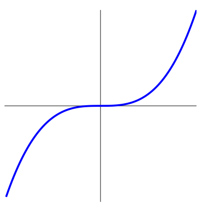
b) The **"Maria Agnesi"** function:

$$f(x) = \frac{1}{1+x^2}$$

c) The **three gorges function**

$$f(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} .$$

Problem 8.4: Match also the following functions with their derivatives. Give short explanations documenting your reasoning in each case.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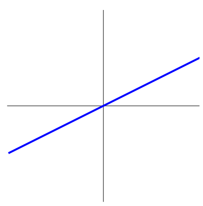
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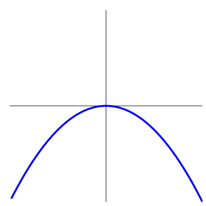
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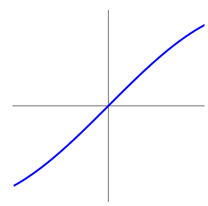
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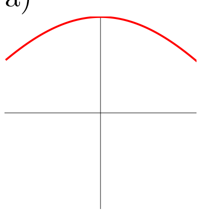
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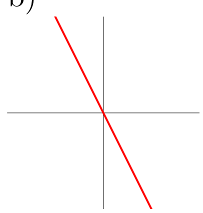
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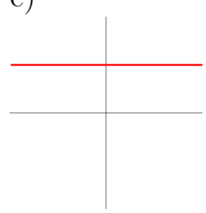
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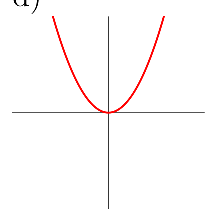
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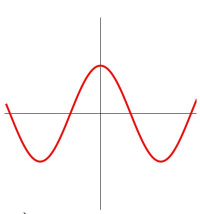
Problem 8.5: Below you the graphs of four different derivative functions $f'(x)$. In each case you are told that $f(0) = 1$. Your task is to draw the function $f(x)$ in each of the cases a),b),c),d). Your picture does not have to be up to scale, but your drawing should display the right features.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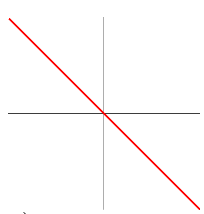
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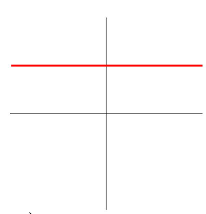
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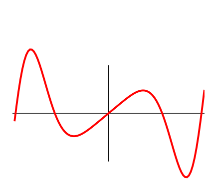
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