

# INTRODUCTION TO CALCULUS

MATH 1A

## Data project 1: Primes

**1.1.** Define the **prime function**  $f(x)$  as the function on positive integers which gives the  $x$ 'th prime. So,

$$f(1) = 2, f(2) = 3, f(3) = 5, f(4) = 7 .$$

By definition, the prime function is **monotone** in the sense that  $f(x + 1) > f(x)$  and more generally  $f(y) > f(x)$  if  $y > x$ . To start this project, make a list of the first 17 primes.

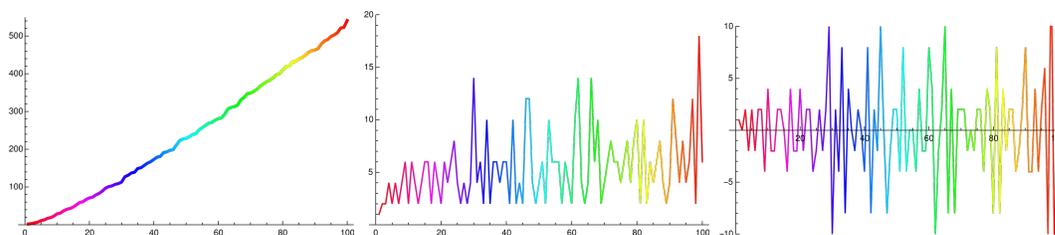


FIGURE 1. Prime function  $f(x)$ , velocity  $f'(x) = f(x + 1) - f(x)$  and acceleration  $f''(x) = f(x + 2) - 2f(x + 1) + f(x)$ .

**1.2.** Define the **derivative**

$$f'(x) = f(x + 1) - f(x) .$$

This measures the increase or decrease between  $x$  and  $x + 1$  and is also known as **rate of change**. For any positive integer  $h$ , the expression

$$\frac{f(x + h) - f(x)}{h}$$

is called the **average rate of change** from  $x$  to  $x + h$ . The name average is justified because

$$\frac{f(x + h) - f(x)}{h} = \frac{f'(x) + f'(x + 1) + \cdots + f'(x + h - 1)}{h}$$

is the average between all the rate of changes. Question 2:

write down the average rate of change equation for  $x = 1$  and  $h = 9$ .

**1.3.** A point  $x$  for which  $f'(x) = 2$  is called a **prime twin**. A big conjecture is that there are infinitely many prime twins. The problem appears too difficult for current mathematics to be solved; but we can find small prime twins, especially with a computer. Third question: find the first 10 prime twins

**1.4.** The second derivative of  $f$  is called the **acceleration**. It is defined as the derivative of the derivative function  $f'$ . We have

$$f''(x) = f'(x+1) - f'(x) = (f(x+2) - f(x)) - (f(x+1) - f(x)) = f(x+2) - 2f(x+1) + f(x).$$

We have  $f''(2) = 0$  because  $f(4) - 2f(3) + f(2) = 7 - 2 * 5 + 3 = 0$ . Fourth problem: find the next two  $x$  for which  $f''(x) = 0$ .

**1.5.** The **fundamental theorem of calculus for data** tells that if  $f'(x) = f(x + 1) - f(x)$ , then

$$f'(a) + f'(a + 1) + f'(a + 2) + \cdots + f'(b - 1) = f(b) - f(a)$$

for any integers  $a, b$ . Verify this for the prime function  $f(x)$  with  $a = 1, b = 10$ . The fundamental theorem will later in the course be written as

$$\int_a^b f'(x)dx = f(b) - f(a).$$

**1.6.** Primes are one of the first things which have been considered in mathematics. Euclid was the first to prove that there are infinitely many primes. His argument was ingenious: assume there would be only finitely many prime data  $p_1, \dots, p_n$ , then the number  $m = p_1 \cdot p_2 \cdots p_n + 1$  has either to be prime or to contain a new prime factor different from the given list; indeed,  $m$  is not divisible by any of the primes. This result is important as it justifies that the **prime function**  $f$  we have been talking about, really exists. In the last part, we want you to look up **Sophie Germain primes**. Fifth problem: Look up the definition Germain primes.

Why was Sophie interested in these primes?

**1.7.** Finally, in problem 6, we want you to look up the largest known prime. There is a large collaborative project going on in which humanity tries to push the largest known prime further. What is the largest known prime today?

What is the name of this prime hunting project?



FIGURE 2. An artistic rendering of Sophie Germain.