

## 5/7/2021: Final Practice E

”By signing, I affirm my awareness of the standards of the Harvard College Honor Code.”

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ... but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

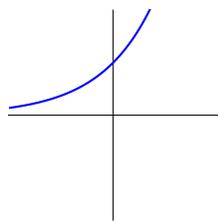
Problem 1) TF questions (20 points) No justifications are needed.

- 1)  T  F The quantum exponential function  $\exp_h(x) = (1 + h)^{x/h}$  satisfies  $D \exp_h(x) = \exp_h(x)$  for  $h > 0$ .
- 2)  T  F The function  $\text{sinc}(x) = \sin(x)/x$  has a critical point at  $x = 0$ .
- 3)  T  F The limit of  $1/\log(1/|x|)$  for  $x \rightarrow 0$  exists.
- 4)  T  F The strawberry theorem tells that for any  $f(x)$ , its anti-derivative  $F(x)$  and  $g(x) = F(x)/x$  the points  $f = g$  are the points where  $g' = 0$ .
- 5)  T  F The function  $f(x) = \tan(x)$  has a vertical asymptote at  $x = \pi/2$ .
- 6)  T  F The function  $x/(1 + x)$  converges to 1 for  $x \rightarrow \infty$  and has therefore a horizontal asymptote.
- 7)  T  F The function  $f(x) = \tan(x)$  is odd: it satisfies  $f(x) = -f(-x)$ .
- 8)  T  F The function  $\sin^3(x)/x^2$  is continuous on the real line.
- 9)  T  F With  $Df(x) = f(x+1) - f(x)$  we have  $D(fg)(x) = Dfg(x+1) + f(x)Dg(x)$ .
- 10)  T  F If  $f$  has a critical point 0 then  $f$  has a minimum or maximum at 0.
- 11)  T  F The limit of  $[\frac{1}{3+h} - \frac{1}{3}]/h$  for  $h \rightarrow 0$  is  $-1/9$ .
- 12)  T  F The function  $(\cos(x) + \sin(3x))/(\sin(4x) + \cos(3x))$  can be integrated using trig substitution.
- 13)  T  F The marginal cost is the anti-derivative of the total cost.
- 14)  T  F The cumulative distribution function is the anti-derivative of the probability density function.
- 15)  T  F The function  $\sqrt{1 - x^2}$  can be integrated by a trig substitution  $x = \cos(u)$ .
- 16)  T  F The integral  $\int_0^1 1/x^2 dx$  is finite.
- 17)  T  F The chain rule tells that  $d/dx f(g(x)) = f'(x)g'(x)$ .
- 18)  T  F For the function  $f(x) = \sin(100x)$  the hull function is constant.
- 19)  T  F The trapezoid method is also called Simpson rule.
- 20)  T  F If  $f''(x) > 0$ , then the curvature of  $f$  is positive.

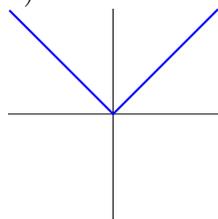
Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their derivatives.

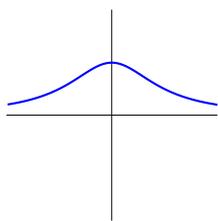
Function	Fill in the numbers 1-8
graph a)	
graph b)	
graph c)	
graph d)	
graph e)	
graph f)	
graph g)	
graph h)	



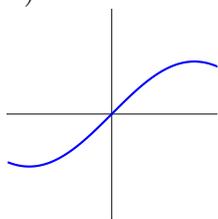
a)



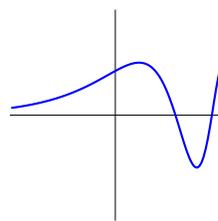
e)



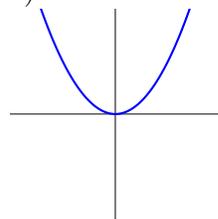
b)



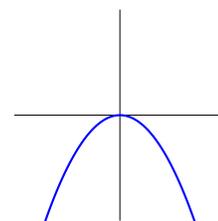
f)



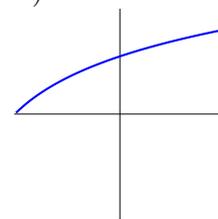
c)



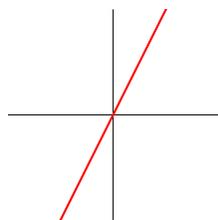
g)



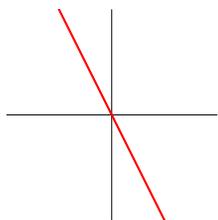
d)



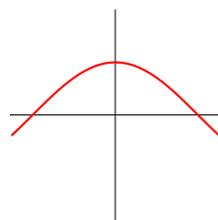
h)



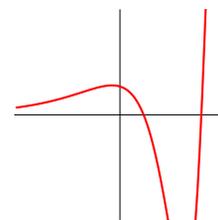
1)



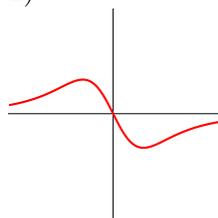
2)



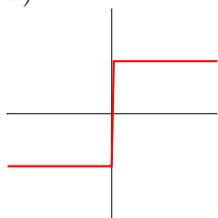
3)



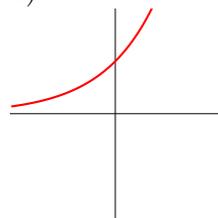
4)



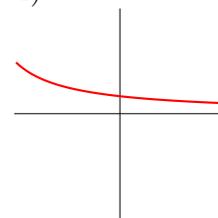
5)



6)



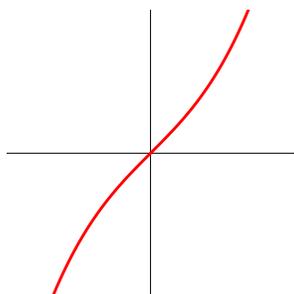
7)



8)

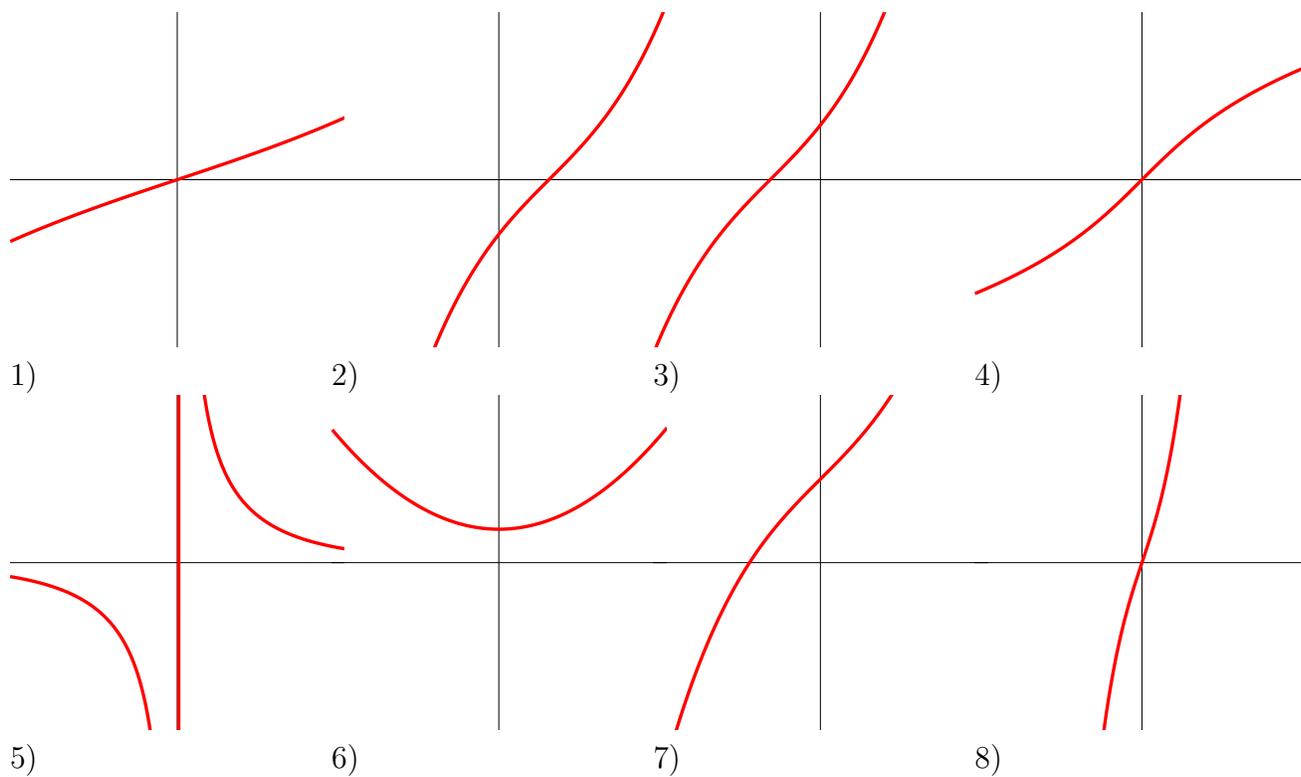
Problem 3) Matching problem (10 points) No justifications are needed.

Here is the graph of a function  $f(x)$ . Match the following modifications



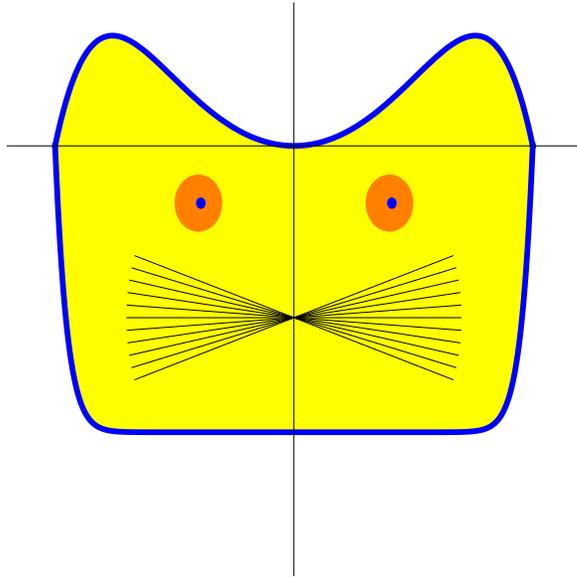
Match the following functions with their graphs.

Function	Fill in 1)-8)
$f(x - 1)$	
$f'(x)$	
$f(x + 1)$	
$f^{-1}(x)$	
$f(x/2)$	
$f(3x)$	
$1/f(x)$	
$f(x) + 1$	



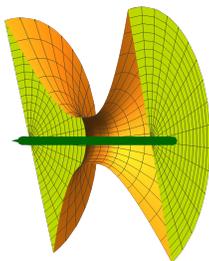
Problem 4) Area computation (10 points)

Find the area of the **cat region** which is the region enclosed by the functions  $f(x) = x^{20} - 1$  and  $g(x) = x^2 - x^6$ . No need to count in the whiskers.



Problem 5) Volume computation (10 points)

We spin the graph of the function  $f(x) = \sqrt{1 + |x|^3}$  around the  $x$  axes and get a solid of revolution. What is the volume of this solid enclosed between  $x = -3$  and  $x = 3$ ? The picture shows half of this sold.



Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals

a)  $\int_{-1}^1 \frac{1}{1+x^2} dx$

b)  $\int_1^2 x^2 + \sqrt{x} \, dx$

c)  $\int_0^{\sqrt{\pi}} \sin(x^2) 2x \, dx.$

d)  $\int_0^1 \log(4 + x) \, dx.$

Problem 7) Extrema (10 points)

a) (7 points) Analyse the local extrema of the function

$$f(x) = \frac{x}{1 + x^2}$$

on the real axes using the second derivative test.

b) (3 points) Are there any global extrema?

Problem 8) Integration by parts (10 points)

a) (5 points) Find the anti-derivative of

$$f(x) = \sin(4x) \cos(3x) .$$

b) (5 points) Find the anti-derivative of

$$f(x) = (x - 1)^2 \sin(1 + x) .$$

Problem 9) Substitution (10 points)

a) (3 points) Find the integral  $\int 3x\sqrt{5x^2 - 5} \, dx.$

b) (3 points) What is the anti-derivative of  $\int \exp(x^2 - x)(4x - 2) \, dx$  ?

c) (4 points) Evaluate the definite integral

$$\int_0^{\pi/2} \sqrt{1 - \cos(x)} \sin(x) \, dx .$$

Problem 10) Partial fractions, Trig substitution (10 points)

a) Solve the integral

$$\int \frac{2 - x + x^2}{(1 - x)(1 + x^2)}$$

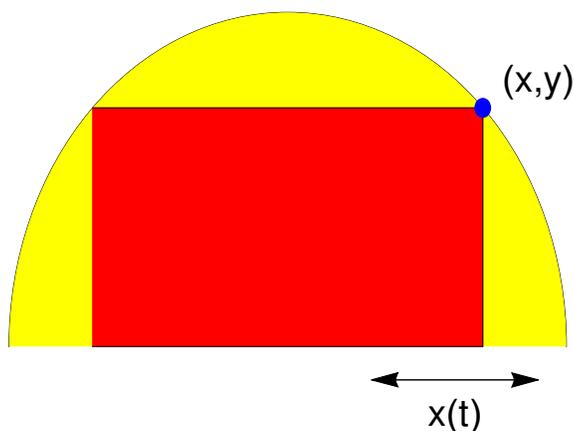
by writing

$$\frac{2 - x + x^2}{(1 - x)(1 + x^2)} = \frac{A}{1 + x^2} + \frac{B}{1 - x}.$$

b) Evaluate the integral  $\int \sqrt{1 - x^2} x \, dx$ .

Problem 11) Differentiation rules (10 points)

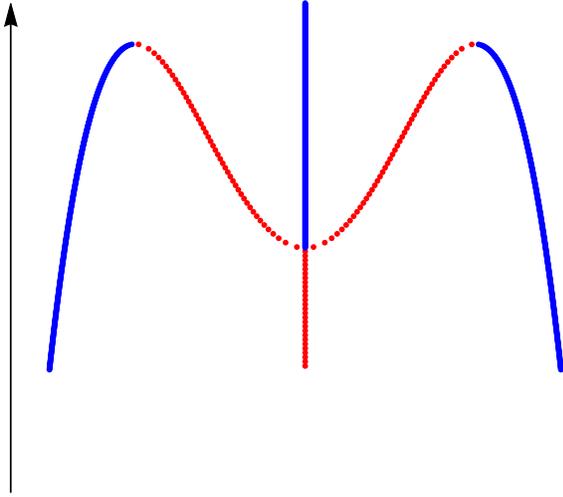
A rectangle with corners at  $(-x, 0)$ ,  $(x, 0)$ ,  $(x, y)$ ,  $(-x, y)$  is inscribed in a half circle  $x^2 + y^2 = 1$  where  $y \geq 0$  is in the upper half plane. We know  $x(t) = \cos(t)$  and  $y(t) = \sin(t)$ . Find the rate of change of the area  $2x(t)y(t)$  of the rectangle.



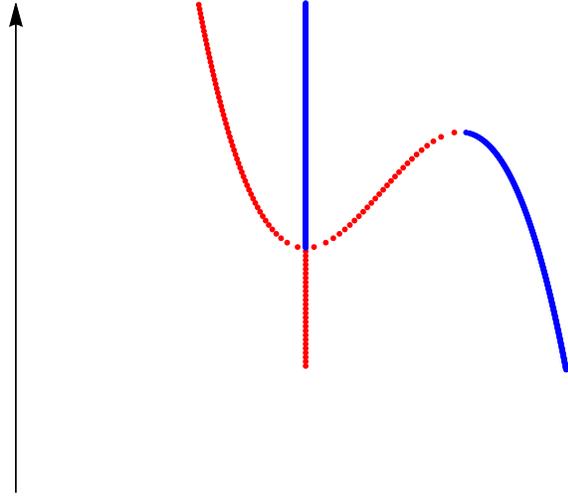
Problem 12) Catastrophes (10 points)

The following two pictures show bifurcation diagrams. The vertical axis is the deformation parameter  $c$ . On the left hand side, we see the bifurcation diagram of the function  $f(x) = x^6 - x^4 + cx^2$ , on the right hand side the bifurcation diagram of the function  $f(x) = x^5 - x^4 + cx^2$ . As done in class and homework, the bolder continuously drawn graphs show the motion of the local minima and the lighter dotted lines show the motion of the local maxima. In both cases, determine the catastrophe for the critical point  $x = 0$ .

c



c



Problem 13) Applications (10 points)

The **Laplace distribution** is a distribution on the entire real line which has the probability density  $f(x) = e^{-|x|}/2$ . The variance of this distribution is the integral

$$\int_{-\infty}^{\infty} x^2 f(x) dx .$$

Find it.

