

5/7/2021: Final Exam

”By signing, I affirm my awareness of the standards of the Harvard College Honor Code.”

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ... but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

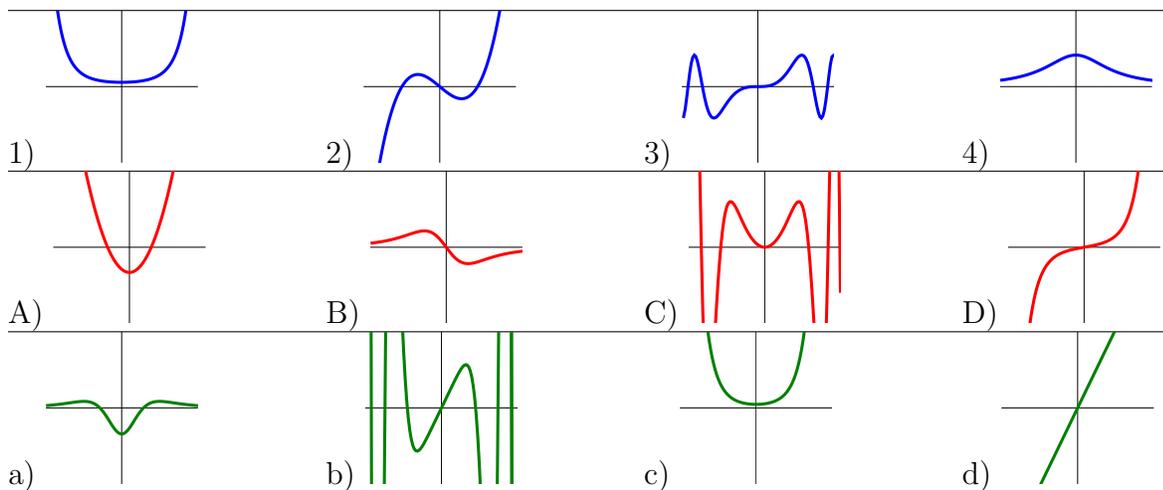
Problem 1) TF questions (20 points). No justifications are needed.

- 1) T F If f is a PDF, then $\int_{-\infty}^{\infty} x^2 f(x) dx$ is called the variance of the distribution.
- 2) T F The function $\log(e^x)$ is defined and continuous for all real numbers x .
- 3) T F There exists $p > 0$ such that $\int_0^{\infty} x^p dx$ exists and is a finite value.
- 4) T F $(e^x)^y = (e^y)^x$ for all real numbers x, y .
- 5) T F $e^{(x^y)} = e^{(y^x)}$ for all real numbers x, y .
- 6) T F The tangent function is continuous everywhere.
- 7) T F There was a time in your life when your age $t \in \mathbb{R}$ in units of years was exactly your height $h \in \mathbb{R}$ in units in meters.
- 8) T F The limit $\lim_{x \rightarrow 0} x \log(|x|)$ is 0.
- 9) T F The Monte-Carlo method is a numerical integration method.
- 10) T F One of the double angle formulas is $\sin(2x) = \frac{(1 - \cos^2(x))}{2}$.
- 11) T F The function $f(x) = x(1 - x^5) + \sin(\pi x)$ has a critical point in $(0, 1)$.
- 12) T F The function $f(x) = \sin(x)/x$ is called the entropy function.
- 13) T F If $Df(x) = f(x+1) - f(x)$ and $Sf(x) = f(0) + f(1) + \dots + f(x-1)$, then $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.
- 14) T F The function $1/\log(2 - |x|)$ is defined and continuous for all real numbers x .
- 15) T F The Newton iteration method allows to find the roots for any continuous function.
- 16) T F The logarithm function $\log(x)$ is monotonically increasing for all $x > 0$.
- 17) T F Applying the Newton step with the function $f(x) = x^2 - 2$ at the point $x = 1$ gives $T(x) = 1 + 1/2 = 3/2$.
- 18) T F The family of functions $f_c(x) = c(x-1)^2 + 3$ experiences a catastrophe at $c = 0$.
- 19) T F If $F(x) = x^2 + 3x$ then $f(x) = 2x + 3$ is called marginal cost.
- 20) T F The gradus suavitatis of the fraction $3/8$ is equal to 6.

Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the name of the functions with their graphs (1-4), with their derivatives (A-D) (middle row) and with the second derivatives (a-d) (last row). If you write on your own paper, please copy the table first.

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$1/(1+x^2)$			
$\sin(x^3)$			
$x^3 - x$			
$\frac{e^{1+x^2}}{20}$			



(5 points) In which cases can we take the limit $x \rightarrow 0$? If there is a limit, enter it in the left column, otherwise cross check the right column. If you write on your own paper, please copy the table first.

Function	The limit is (if it exists)	Cross check if not existing
$\frac{\sin(17x)}{\sin(23x)}$		
$-x \log 3x $		
$\frac{\sin(x^2)}{\sin^2(x)}$		
$\log 5x / \log 7x $		
$\arctan(x) / \tan(x)$		
$\frac{\cos(x)+1}{x^2}$		

Problem 3) Short answer problem (10 points). No justifications are needed.

a) (3 points) Complete the following table of probability distributions and cumulative distribution functions.

PDF	PDF supported on	CDF on that interval
e^{-x}	$[0, \infty)$	
	$(-\pi/2, \pi/2)$	$\frac{\arcsin(x)}{\pi} + \frac{1}{2}$
	$(-\infty, \infty)$	$\frac{\arctan(x)}{\pi} + \frac{1}{2}$

b) (3 points) We integrate $\int_0^1 x dx$ numerically, using only $n = 1$ interval and compare three different integration methods. What values do we get in each case?

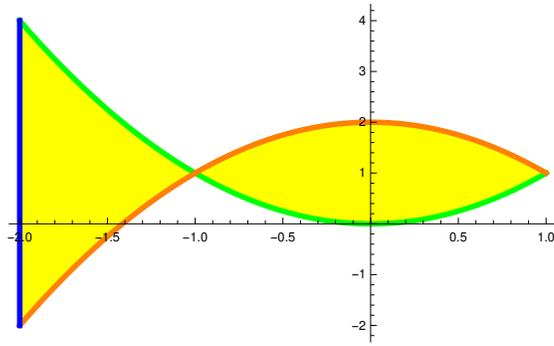
Integration method	Which value does the method give?
Trapezoid method	
Simpson Method	
Simpson 3/8 Method	

c) (2 points) If f is the marginal cost and F the total cost and g the average cost. What is the definition of the **break even point** in this context?

d) (2 points) What theorem is responsible for the fact that there is a point on earth such that the temperature on P and its anti-pod point Q are exactly the same?

Problem 4) Area computation (10 points)

What is the area of the **fish shape** region bordered by the curves $y = x^2$, $y = 2 - x^2$, $x = -2$? We stress that we want the **area** of the region and **not** the signed area of the region.

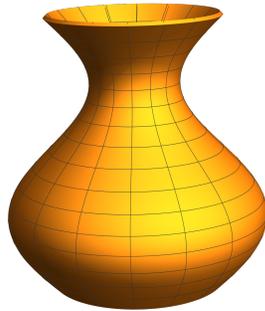


Problem 5) Volume computation (10 points)

Compute the volume of the **vase** which has at height z a **circular cross sections of radius**

$$r(z) = 2 + \sin(z)$$

and where the height z is in the interval $0 \leq z \leq 2\pi$.



Problem 6) Improper integrals (10 points)

You see here five improper integrals. Pick the ones which converge and compute their value. For the others, state that the integral does not exist and give a short explanation.

a) (2 points) $\int_0^1 x^{-6} dx$

b) (2 points) $\int_0^1 x^{-1/6} dx$

c) (2 points) $\int_1^\infty x^{-6} dx$

d) (2 points) $\int_1^\infty x^{-1/6} dx$

e) (2 points) $\int_0^\infty x^{-6} dx$.

Problem 7) Extrema (10 points)

A **trapezoid** of height $2x$, bottom length x and top length $(4 - 2x^2)$ has area

$$f(x) = x(x + (4 - 2x^2)) = 4x + x^2 - 2x^3 .$$

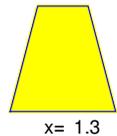
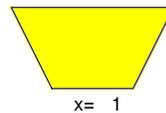
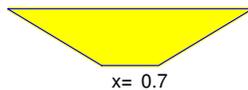
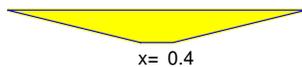
The figure below illustrates the situation for some x values.

a) (3 points) For which x does f have a local maximum? Use the second derivative test to check.

b) (3 points) On which interval $[a, b]$ does the problem make sense (height, lengths and area can not be negative)? What are the function values on the boundary?

c) (2 points) There a theorem which assures that $f(x)$ has a global maximum on that interval $[a, b]$? What is the name of the theorem?

d) (2 points) What is the global minimum of $f(x)$ on the interval of consideration?



Problem 8) Integration by parts (10 points)

a) (5 points) Compute the following anti-derivative:

$$\int (x - 2)^4 e^{x/2} dx .$$

b) (5 points) And now have some fun riding the “merry go round” for the following integral

$$\int e^x \sin(x) dx .$$

Problem 9) Substitution/chain rule (10 points)

a) (5 points)

$$\int \cos(\cos(\cos(x))) \sin(\cos(x)) \sin(x) dx .$$

b) (5 points)

$$\int \frac{2 \log(\log(\log(x)))}{\log(\log(x)) \log(x) x} dx .$$

Problem 10) Advanced integration (10 points)

a) (5 points) Integrate

$$\int \frac{1}{(x+8)(x+4)(x+2)} dx .$$

b) (5 points) Use the magic **trig substitution box** to find the anti-derivative:

$$\int \frac{1}{\cos(x)} + \frac{1}{\sin(x)} dx .$$

Here is the magic box:

$$\begin{aligned} u &= \tan(x/2) \\ dx &= \frac{2du}{1+u^2} \\ \sin(x) &= \frac{2u}{1+u^2} \\ \cos(x) &= \frac{1-u^2}{1+u^2} \end{aligned}$$

Problem 11) PDF's and CDF's. (10 points)

We look at two functions which are defined to be zero outside the interval $[0, \pi]$ and which are given on $0 \leq x \leq \pi$ as

$$f(x) = \frac{\sin(x)}{2}$$

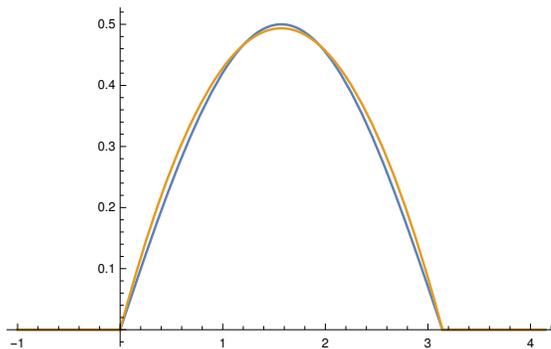
and

$$g(x) = \frac{x(\pi - x)}{5} .$$

Their graphs are very close to each other, as you can see in the picture.

a) (5 points) One of these functions produces a probability distribution. Which one? Justify your answer.

b) (5 points) The other function needs to be multiplied with a constant c to become a probability distribution. What is this constant?



Problem 12) Which integration method? (10 points)

a) (5 points) Determine the value of the definite integral

$$\int_0^{\pi/2} e^{\sin(x)} \cos(x) dx .$$

b) (5 points) Find the anti-derivative

$$\int 3\sqrt{1-x^2} + \frac{4}{\sqrt{1-x^2}} dx .$$

Problem 13) Applications (10 points) Only answers are needed

a) (2 points) What is the expectation $\int xf(x) dx$ of the probability distribution function $f(x) = e^{-|x|}/2$ which is a function defined for all real x . [We only need a number.]

b) (2 points) What is the inverse function of the (fictional) **Stockhausen Midi function** $f(x) = 440 \cdot 5^{(x-25)/5}$? This function gives the midi number x as a function of the frequency f .

c) (2 points) Which digit among $0, 1, \dots, 8, 9$ is the **least frequent** in **Benford's distribution law**? [You only need to provide the digit]

d) (2 points) The family of functions $f_c(x) = x^4 - cx^2 + x^2$ experiences a **catastrophe**. What is the parameter c for which this happens? [You only need to find that parameter.]

e) (2 points) If $(-2, -7)$ and $(2, 7)$ are two data points. What is the best linear fit $y = ax + b$ which passes through the two points?